THE AMERICAN MATHEMATICAL MONTHLY

The endless torch race first began
  Who knoweth where? Who knoweth when?
. . . . . .
The runners give from hand to hand,—
In lines of flame from land to land;
  And still again, and yet again,
They follow straight the soul's command,—
Though but in part they understand,—
And still pass on the deathless brand
  From man to man—and we are men!

WILLIAM ADAMS SLADE.

THE APRIL MEETING OF THE IOWA SECTION.

The Iowa Section of the Mathematical Association of America met for its fourth regular meeting April 26, 1919, in connection with the Iowa Academy of Science, at the Iowa State Teachers College, Cedar Falls, Iowa. There were present as members of the Association: L. M. Coffin, Coe College; I. S. Condit, Iowa State Teachers College; F. M. McGaw, Cornell College; J. F. Reilly, University of Iowa; H. L. Rietz, University of Iowa; Maria M. Roberts, Iowa State College; F. M. Weida, University of Iowa; and C. W. Wester, Iowa State Teachers College.

The following papers were read and discussed:

"The teaching of a first course in mathematics." By Professor H. L. Rietz.

"A course in arithmetic." By Professor I. S. Condit.

"Effect of delaying algebra until the tenth grade." By Miss Lida Pittman, Fort Dodge (by invitation).

"An example of curve fitting." By G. W. Snedecor, Ames (by invitation).

"Outlines of a course in trigonometry." By Professor J. F. Reilly.

"Some analogies between algebraic equations and linear differential equations." By Peter Luteyn, State Teachers College (by invitation).

The following were elected officers for the ensuing year: Chairman, I. S. CONDIT; Vice Chairman, F. M. MCGAW; Secretary-Treasurer, L. M. COFFIN.

C. W. WESTER, *Secretary-Treasurer.*

SOME EXTENSIONS OF THE WORK OF PAPPUS AND STEINER ON TANGENT CIRCLES.

By J. H. WEAVER, Ohio State University.

Introduction. The figure of three mutually tangent semicircles with their centers in the same straight line was known among the Greeks as the "Shoemaker's Knife" (ἄρβηλος). A few of the properties of the figure are found in the works of Archimedes.[1] Others occur in the Collection of Pappus.[2] After the Greek period we find no work done on the problem until Steiner generalized the results of Pappus and added several others dealing with the perspective properties of the figure.[3] Later Sir Thomas Muir added a theorem giving formulæ for various sets of radii involved.[4] Habicht has discussed some of the properties of elliptic functions connected with the figure[5] while M. G. Fontené has generalized certain formulæ arising from sets of tangent circles.[6]

In the present paper formulæ for the radii of certain sets of circles are developed and used to build up several types of infinite series which may be summed geometrically. Then some general properties of tangents and normals to conics associated with three mutually tangent circles are set forth.[7] These properties lead to a quadrangular-quadrilateral configuration and incidentally furnish some methods for constructing conics. And finally some theorems connected with centers of perspectivity of the various sets of circles are proved.

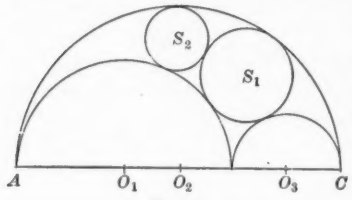


FIG. 1.

1. General Considerations. Let there be two circles tangent internally, with centers O_1 and O_2 and let a circle with center S_n ($n = 1, 2, \cdots$) (Fig. 1) be tangent to these. Then S_n lies on an ellipse with foci O_1 and O_2. If we take the midpoint of O_1O_2 as origin and O_1O_2 as the x-axis, the equation of the ellipse will be

$$\frac{4x^2}{(r_1+r_2)^2}+\frac{y^2}{r_1r_2}=1, \tag{1}$$

where r_1 and r_2 ($r_2 > r_1$) are the radii of the circles (O_1) and (O_2) respectively.

[1] *Works of Archimedes*, ed. Heath, Cambridge, 1897, Lemmas, 4–6.

[2] *Collectio*, ed. Hultsch., Berlin, 1876–8, Vol. I, pp. 209 and ff.

[3] Steiner, *Gesammelte Werke*, Berlin, 1881, Vol. 1, pp. 47–76.

[4] *Proceedings of Edinburgh Math. Soc.*, Vol. 3, p. 119. In the same volume, pages 2–11, J. S. Mackay has collected some of the simpler theorems connected with the problem.

[5] Konrad Habicht, *Die Steinerschen Kreisreihen*, Berne, 1904, 35 pp. In this work are found extensive references bearing on the subject.

[6] "Sur les cercles de Pappus," *Nouvelles Annales de Mathématiques* (4), tome 1918, pp. 383–90.

[7] The center of a circle tangent to two given circles lies on a conic having the centers of the two given circles as foci. This is, of course, equivalent to the definition that the sum or difference of the focal radii is constant. I have called such conics "associated" conics.

[8] In what follows circles will be designated by their centers in brackets.

Let ρ_n be the radius of (S_n) and let the coördinates of the point S_n be x_n and y_n. From a fundamental property of the ellipse we have

$$r_1 + \rho_n = a + ex_n, \tag{2}$$

where $2a = r_1 + r_2$, and

$$e = \frac{r_2 - r_1}{r_2 + r_1}.$$

Pappus has shown that if another circle (S_{n+1}) with radius ρ_{n+1}, center at point x_{n+1}, y_{n+1} and coming after (S_n) in the positive direction around the circles, is tangent to (S_n), the following relation holds

$$\frac{y_n + 2\rho_n}{\rho_n} = \frac{y_{n+1}}{\rho_{n+1}}$$

or

$$\frac{y_n}{\rho_n} = \frac{y_{n-1}}{\rho_{n-1}} + 2 = \frac{y_1}{\rho_1} + 2(n-1). \tag{3}$$[1]

2. Formulæ arising from the figure of three mutually tangent circles with their centers in the same straight line. Let there be three mutually tangent circles (O_1, O_2) and (O_3) having their centers in the same straight line and radii r_1, r_2, and r_3 respectively (Fig. 1). Then let a series of circles (S_n) be drawn tangent to (O_1) and (O_2), the first circle in the series being also tangent to (O_3) and each of the others tangent to the one preceding it in the series. There are two other such sets of tangent circles. The set tangent to (O_2) and (O_3) we will designate as (S_n'), and the set tangent to (O_1) and (O_3) as (S_n''). Let the radii of the various sets be ρ_n, ρ_n' and ρ_n'' respectively, and the coördinates of the centers be x_n, y_n, x_n', y_n' and x_n'', y_n'' respectively. We will now consider the set (S_n).

By means of equations (1), (2) and (3) and the use of induction we have in this particular case, since the y-coördinate of $O_3 = 0$

$$\rho_n = \frac{r_1 r_2 r_3}{n^2 r_3^2 + r_1 r_3 + r_1^2}. \tag{4}$$

This result is arrived at by Muir and Fontené by different methods.[2] Also from equations (2) and (4)

$$x_{n-1} - x_n = \frac{(2n-1) r_3^2 r_1 r_2 (r_1 + r_2)}{[(n-1)^2 r_3^2 + r_1 r_3 + r_1^2][n^2 r_3^2 + r_1 r_3 + r_1^2]} = i_n, \text{ say.} \tag{5}$$

From the geometric properties of the figure

$$\sum_{n=1}^{\infty} \rho_n \tag{6}$$

is a convergent series, and if r_3 approaches the limit 0, then (6) approaches the value $\pi r_2/2$ but is 0 at the limit.

[1] Pappus, *Collectio*, p. 224.
[2] See Introduction.

Also

$$\sum_{n=1}^{\infty} i_n = 2r_1 + r_3.$$

If we define i_n as the nth intercept of the series (S_n) (i_1, projection of O_3S_1 on AC), then (4) and (5) are the formulæ for the nth radius and intercept in the series (S_n). An interchange of r_1 and r_3 will give the corresponding formulæ for the series (S_n'), while an interchange of r_2 and r_3 with r_2 considered negative will give the corresponding formulæ for the set (S_n'').

If $r_2 = 2r_1$ we have the special case

$$\sum_{n=1}^{\infty} \rho_n'' = r_2/2.$$

We will now establish the following theorem.

THEOREM: If two circles (O_1) and (O_2) are tangent internally, and a circle (S) is drawn tangent to these two, such that SO_i $(i = 1, 2)$ is perpendicular to O_1O_2 then it is possible to draw a circle (S') tangent to (O_1), (O_2) and (S) such that the four centers S, S', O_1 and O_2 determine a rectangle.

Proof: Let SO_2 be perpendicular to O_1O_2, and let the coördinates of S and S' be x, y and x', y' respectively and let the radii be ρ and ρ'. Then

$$x = \frac{r_2 - r_1}{2}, \qquad y = \frac{2r_1r_2}{r_1 + r_2}, \qquad \rho = \frac{r_2(r_2 - r_1)}{r_1 + r_2}, \tag{6}$$

and by virtue of equations (1), (2) and (3)

$$y' = y \text{ and } x' = -x,$$

which proves the theorem.

THEOREM: If in the series (S_n), the points S_n, S_{n+1}, O_1 and O_2 determine a rectangle, then $r_1 = nr_3$.

Proof: Equate the values of ρ given in equations (4) and (6).

Let the foot of the perpendicular from S_n to O_1O_2 be P_n.

Let angle $P_nS_nS_{n+1} = \angle B_n$.

Then if $r_1 = kr_3$ (k an integer or a rational fraction)

$$\tan B_n = \frac{(2k + 1)(2n + 1)}{2(n^2 + n - k - k^2)} \tag{7}$$

and the slopes of the lines of successive centers of the series (S_n) are all rational. Moreover if in (7) k is an integer, $B_k = \pi/2$, and

$$\sum_{n=k+1}^{\infty} (B_{n-1} - B_n) = B_k - \lim_{n=\infty} B_n = \pi/2.$$

From the identity $(\rho_n + \rho_{n+1})^2 = i_{n+1}^2 + (y_n - y_{n+1})^2$ we get by using equa-

tions (3), (4) and (5) the equation

$$[(2n^2+2n+1)r_3^2+2r_1r_3+2r_1^2]^2$$
$$=[(2n^2+2n)r_3^2-2r_1r_3-2r_1^2]^2+[(2n+1)r_3(r_2+r_1)]^2$$

giving a triply infinite set of rational right triangles.

3. Formulæ arising from three mutually tangent circles, one of which is tangent to the line of centers of the other two. Let there be two circles (O_1) and (O_2) (Fig. 2) tangent internally at A and let a series of circles be drawn tangent to these, the first one in the series being tangent to the line O_1O_2 and each of the others tangent to the one preceding it in the series. Let the radius of (O_1) be r_1 and of (O_2) be r_2, and let the centers of the circles be S_n. Then from equations (1), (2) and (3), since $y_1=\rho_1$ we get by induction

$$\rho_n=\frac{4r_1r_2(r_2-r_1)}{4(n^2-n)(r_2-r_1)^2+(r_2+r_1)^2} \tag{8}$$

$$i_n=\frac{32(n-1)r_1r_2(r_2-r_1)^2(r_2+r_1)}{[4(n^2-n)(r_2-r_1)^2+(r_2+r_1)^2][(4n^2-12n+8)(r_2-r_1)^2+(r_2+r_1)^2]} \tag{9}$$

where for i_n $n\geqq 2$.

Here

$$\sum_{n=2}^{\infty} i_n = 2r_2 - i_1{}^{1}$$

and if r_1 approaches r_2

$$\sum_{n=2}^{\infty}\rho_n$$

approaches $\pi r_2/2$.

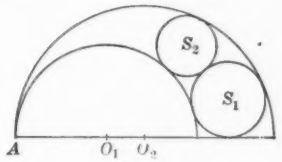


FIG. 2.

THEOREM: If in this series the points S_n, S_{n+1}, O_1 and O_2 determine a rectangle then

$$r_2=\frac{2n+1}{2n-1}\cdot r_1.$$

Let

$$r_2=\frac{2k+1}{2k-1}\cdot r_1.$$

Then

$$\tan B_n=\frac{2nk}{n^2-k^2}.$$

Moreover,

$$\sum_{n=k+1}^{\infty}(B_{n-1}-B_n)=B_k-\lim_{n=\infty}B_n=\pi/2.$$

Also the equation $(\rho_n+\rho_{n+1})^2=(i_{n+1})^2+(y_n-y_{n+1})^2$ gives the triply infinite set of rational right triangles[2]

$$[4n^2(r_2-r_1)^2+(r_2+r_1)^2]^2=[4n^2(r_2-r_1)^2-(r_2+r_1)^2]^2+[4n(r_2^2-r_1^2)]^2.$$

[1] The distance from the point of contact of S_1 with the diameter AO_2 to the end of this diameter, on the side opposite from the point O_2,, is taken as i_1.—Editor

[2] This result is but a special case of a rational right triangle with sides u^2+v^2, u^2-v^2, and $2uv$.—Editor.

4. Formulæ arising from a series of tangent circles, tangent to two given circles, the first circle in the series being tangent to a line tangent to the smaller circle and perpendicular to the line of centers. Let there be two circles (O_1) and (O_2) tangent internally at A and let $(O_1) < (O_2)$ and let the tangent to (O_1) perpendicular to O_1O_2 be drawn and let a series of tangent circles be drawn tangent to (O_1) and (O_2), the first circle in the series being also tangent to the perpendicular just drawn (Fig. 3).

Then from Pappus, Book IV., lemma XIX, we have

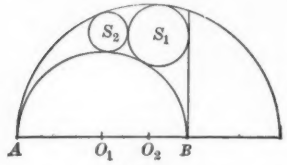


FIG. 3.

$$\frac{r_2 - r_1}{r_2} = \frac{4\rho_1^2}{y_1^2}.$$

Let

$$\frac{r_2 - r_1}{r_2} = m^2,$$

then

$$2\rho_1 = my_1. \tag{10}$$

Using equations (1), (2), (3) and (10) we have by induction

$$\rho_n = \frac{r_1 m^2}{(n-1)^2 m^4 + 2(n-1)m^3 + 1},$$

$$i_n = \frac{r_1 m^3 (2 - m^2)[(2n-3)m + 2]}{[(n-2)^2 m^4 + 2(n-2)m^3 + 1][(n-1)^2 m^4 + 2(n-1)m^3 + 1]}.$$

If r_1 approaches r_2, the sum $\sum_{n=1}^{\infty} \rho_n$ approaches $r_2 \cdot \pi/2$ but is zero at the limit. Also $\sum_{n=1}^{\infty} i_n = 2r_1$, if $i_1 = \rho_1$. Let $\angle B_n = \pi/2$. Then since

$$\sin B_n = \frac{i_{n+1}}{\rho_n + \rho_{n+1}}$$

we have the relation

$$n = \frac{1 - m}{m^2},$$

and if $n = 1$, $r_1 : r_2 =$ side of decagon inscribed in a circle of unit radius. Here also as in sections (2) and (3) we may obtain an equation

$$[(2n^2 - 2n + 1)m^4 + (4n - 2)m^3 + 2]^2$$

$$= [2n^2 - 2n)m^4 + (4n - 2)m^3 + 4m^2 - 2]^2 + [m(2 - m^2)((2n - 1)m + 2)]^2,$$

which gives a doubly infinite set of rational right triangles.

5. Formulæ arising from a series of tangent circles tangent to a given circle and a given straight line. Let (Cf. figure 3) the series of circles (S_n) be tangent to the perpendicular at B and to O_2, (S_1) being also tangent to O_1.

The centers S_n lie on a parabola with vertex O_1 and focus O_2. Using O_1 as origin the equation of the parabola is

$$y^2 = 4(r_2 - r_1)x. \tag{11}$$

Moreover

$$\rho_n = r_1 - x_n \tag{12}$$

and

$$(y_n - y_{n-1})^2 = (\rho_n + \rho_{n-1})^2 - (x_n - x_{n-1})^2. \tag{13}$$

By a substitution from (12) equation (13) reduces to

$$(y_n - y_{n-1})^2 = 4\rho_n\rho_{n-1}. \tag{14}$$

Let $r_1 = \lambda r_2$. Let D_n denote the sum of the odd-numbered terms in the expansion of $(1 + \sqrt{\lambda})^n$: and N_n the sum of the even-numbered terms, that is

$$D_n + N_n = (1 + \sqrt{\lambda})^n,$$

$$D_n - N_n = (1 - \sqrt{\lambda})^n.$$

Then using equations (11), (12) and (14) and induction

$$\rho_n = \left(1 - \frac{N_n^2}{D_n^2}\right) r_1,$$

$$y_n = \frac{2N_n}{D_n}\sqrt{r_1(r_2 - r_1)}. \tag{15}$$

From (15)

$$\frac{2\sqrt{(1-\lambda)r_1r_2} - y_n}{2\sqrt{(1-\lambda)r_1r_2} + y_n} = \frac{(\sqrt{r_2} - \sqrt{r_1})^n}{(\sqrt{r_2} + \sqrt{r_1})^n}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{2\sqrt{(1-\lambda)r_1r_2} - y_n}{2\sqrt{(1-\lambda)r_1r_2} + y_n} = \frac{\sqrt{r_2} - \sqrt{r_1}}{2\sqrt{r_1}}.$$

Also we have

$$\sum_{n=1}^{\infty} (y_n - y_{n-1}) = 2\sqrt{r_1(r_2 - r_1)}.$$

6. Some properties of conics associated with three mutually tangent circles. Let there be two circles (O_1) and (O_2) tangent internally at A (Fig. 4) and let the radii of these circles be r_1 and r_2 respectively, and let the radius of a circle tangent to these two and having its center on the line O_1O_2 be r_3 and let its center be O_3. Let O be the center of any circle tangent to (O_1) and (O_2) and let r be its radius.

The conic associated with (O) and (O_2) is an ellipse with the points O and O_2 as foci, and passing through O_1: the conic associated with (O) and (O_1) is a hyperbola passing through O_2 and having O and O_1 as foci. Likewise we will have an ellipse passing through O and having O_1 and O_2 as foci. Draw from A the line AT tangent to the circles (O_1) and (O_2). Then with the three circles $(O_1,)$ (O_2) and (O) the straight line AT there will be associated four parabolas[1] two of which pass through A, one through O_1 and the fourth through O_2.

[1] See article "Some Properties of a Straight Line and Circle and their Associated Parabolas," *Annals of Math.*, second series, Vol. 19, pp. 174–5. Also "Some properties of circles and related conics," *Annals of Math*, second series, vol. 20, pp. 279–280.

Call the conic through O_2, H_2, the one through O_1, E_1 and the one through O and O_3, E_3, and the parabolas through O_1 and O_2, P_1 and P_2 respectively.

With this notation the following may be readily proved analytically:

THEOREM: The normals to E_1 and H_2 at the points O_1 and O_2 intersect on a line through O_3 perpendicular to O_1O_2.

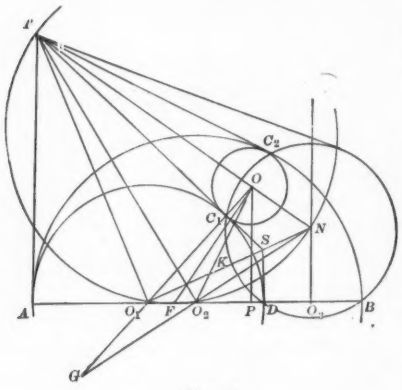


FIG. 4.

THEOREM: The tangents to H_2 and E_1 at the points O_2 and O_1 intersect on the line AT.

THEOREM: If two circles are tangent internally, and any circle is tangent to these two, there are associated with these circles three non-degenerate conics, six points of which are the three points of contact of the three circles and the three centers of the three circles, and the six tangents to the conics at these six points pass through a common point.

Proof: Let the normal to E_1 at O_1 and to H_2 at O_2 intersect at N, and let the intersection of the tangents at these points be T. Let the point of contact of (O) and (O_1) be C_1 and of (O) and (O_2) be C_2. Then the angle AO_1C_1 is bisected by O_1T. Therefore a line from T to C_1 will be tangent to (O_1) and (O). Moreover since T and N are ex-centers of the triangle OO_1O_2, T, N, and O are collinear. It is also evident that the tangent to (O_2) at C_2 will pass through T. We have therefore the six lines TA, TO, TO_1, TC_2 TO_2, and TC_1, and these lines are the six tangents to E_3, E_1 and H_2 at the points A, O, O_1, C_2, O_2, C_1.

THEOREM: E_1 and P_1 have the same normal at O_1.

Proof: Since O is the focus of P_1 and the axis is parallel to O_1O_2, then the bisector of the angle OO_1O_2 will be normal to P_1. But this is also normal to E_1 because O and O_2 are the foci of E_1.

THEOREM: The three axes of the three non-degenerate conics associated with three tangent circles, and the three normals at the centers of the circles, meet in points that are collinear.

Proof: Let the normal and axis of E_3 intersect in F, the normal and axis of H_2 intersect in G and the normal and axis of E_1 in K. Then since we have the triangle O_1O_2O and the two bisectors of two interior angles and the bisector of the opposite exterior angle, the points F, K and G are collinear.

The right angles formed by the tangents and normals at O_1 and O_2 are inscribed in a semicircle with TN as diameter. Call this circle C_t. Steiner has pointed out the fact that D, B, C_1 and C_2 are points of a circle C_n with center N.[1] It is then evident that the tangents to C_n at its points of intersection with C_t pass through T. We then have two sets of coaxial circles C_n and C_t, the centers

[1] See reference to Steiner in Introduction.

of one being on a line through O_3 perpendicular to O_1O_2 and the diameters of the other being segments of tangents to E_3 cut off by the tangents at A and O_3.[1] It should also be noted in this connection that the point T is the pole of the line drawn from the point of tangency of any two of the circles to the center of the third circle with respect to the conic passing through that center.

Also if there is drawn at D a line perpendicular to O_1O_2 and TC_1 is produced intersecting this line in S, then N, S and O_1 are collinear.

THEOREM: If three circles are mutually tangent and tangents and normals be drawn to the three associated conics at the points of contact and the centers of the three circles, and if the normals of two of the conics be chosen, these will intersect by twos on a tangent to the third conic.

Proof: Consider the lines O_1N, O_2N, OO_1, OO_2. These intersect in the points O and N which are on the tangent to E_3.

THEOREM: The axes and normals to two of the conics, together with the tangents to these two conics drawn at the centers of the circles determine two perspective triangles whose center of perspectivity is the intersection of the axis and normal to the third conic.

Proof: Consider the lines O_1N, O_2N, OO_1, OO_2, O_1T, O_2T. These intersect by twos on the line TN. They may therefore be considered as the sides of two perspective triangles. Let the corresponding sides be

$$O_1T, \quad O_2T,$$
$$O_2N, \quad O_1N,$$
$$OO_1, \quad OO_2.$$

These determine the perspective triangles $A_{12}A_{13}A_{23}$ and $B_{12}B_{13}B_{23}$ and the center of perspectivity is the point F on O_1O_2 (see Fig. 4 where F is marked). But this point is also on the normal OB_{12}.

Corollary: There will be three such sets of perspective triangles and the centers of perspective will be collinear (second theorem before the last).

THEOREM: N and T are double points of an involution, of which O and the point (Y), where TN intersects O_1O_2, are a conjugate pair, and therefore O and Y are inverse points with respect to the circle C_t.

The proof of this theorem follows immediately from a consideration of the quadrangle $O_1A_{12}O_2B_{12}$.

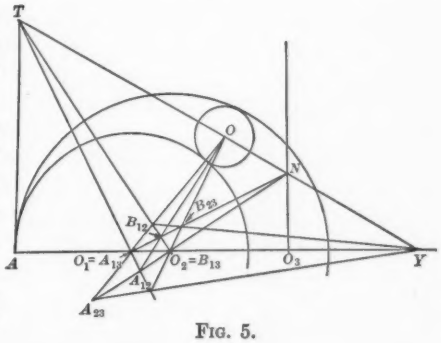


FIG. 5.

The following theorems are also evident.

THEOREM: If the three axes and the three normals to the three associated

[1] See *Conics* of Apollonius, Book III, prop. 45 (Heath's ed., p. 114).

conics be drawn the axis and normal to each conic being taken as corresponding sides, they form two perspective triangles with T as center of perspective.

THEOREM: The four axes of perspective of the three circles and the six lines, three of which are normals and the other three are the tangents to the three conics at the three centers of the three circles, determine a quadrangular-quadrilateral configuration, whose diagonal triangle is the triangle determined by the three centers of the three circles.

In connection with the above discussion it should be noted that it furnishes a method for constructing points on a conic. For let O_1, O_2 and O_3 be any three points on a line, and let the perpendicular be drawn at O_3 and let N be any point in the perpendicular. Let O_1 and O_2 be points such that we have the order $O_1O_2O_3$ or $O_2O_1O_3$. Then from N draw lines to O_1 and O_2, making the angles NO_1O_3 and NO_2O_3, and draw from O_1 and O_2 the lines O_1O and O_2O such that

$$\angle\, NO_1O_3 = \angle\, OO_1N,$$

$$\angle\, NO_2O_3 = \angle\, OO_2N.$$

Then the point O is on an ellipse. If we have the order $O_1O_3O_2$, O will be on a hyperbola, and if O_1 or O_2 is at infinity we have a parabola. And in each instance O_1 and O_2 are foci of the conic. This also gives a method for establishing a (1, 1) correspondence between the points of a conic and the points of a straight line.

7. Some Projective Properties of the Figure in Section 2. Let A and C be the ends of the diameter O_1O_2 of the circle (O_2) (Fig. 1), and let there be drawn from A lines to S_n and from C lines to S_n', and let C and C' be the angles that these lines make with AC. Then

$$\tan C = \frac{2nr_3}{r_1 + r_2}, \tag{16}$$

$$\tan C' = \frac{2nr_1}{r_2 + r_3}. \tag{17}$$

By means of equations (16) and (17) we may find the equations of the lines AS_n and CS_n', a solution of which reveals the fact that the line AS_n and the line CS_n' intersect on a line through S_1 perpendicular to AC in points whose ordinates are $2\rho_n$.[1]

By very simple analytical considerations we may prove the following

THEOREM: The triangles $S_{n+1}S_nS_{n-1}$ and $S_{n+1}'S_n'S_{n-1}'$ are perspective and their center of perspective is the external center of perspective of the circles O_1 and O_3.

THEOREM: The locus of the point of contact of two tangent circles which are tangent to two given tangent circles (internally tangent) is a circle whose center is the center of perspective of the two given circles and whose radius is the harmonic mean between the radii of the two given circles.

[1] In this connection see Steiner, p. 69 and ff.

Proof: The center of perspective, P_3, of the two given circles, O_1 and O_2, has the same power with respect to all circles tangent to these two in a given way. Therefore, the locus of the point of contact of any two such circles which are tangent to each other is a circle orthogonal to them all.

THEOREM: The circle with P_3 as center and P_3A as radius cuts every C_n orthogonally.

Proof: Let there be drawn with T as center and TA as radius a circle. This will pass through C_1 and C_2. Therefore $C_1C_2P_3$ will be the radical axis of the circle just drawn and C_n. The circle with P_3 as center and P_3A as radius is orthogonal to the circle with center T. It is therefore orthogonal to every C_n.

SOME VANISHING AGGREGATES CONNECTED WITH CIRCULANTS.

By W. H. METZLER, Syracuse University.

In the course of certain investigations on Lagrange's Equation for circulants[1] by Dr. Muir[2] in 1912 attention was called to the vanishing aggregate:

$$\begin{vmatrix} 1 & b & c & d \\ 1 & c & d & e \\ 1 & d & e & a \\ 1 & e & a & b \end{vmatrix} + \begin{vmatrix} 1 & b & c & e \\ 1 & c & d & a \\ 1 & d & e & b \\ 1 & a & b & d \end{vmatrix} + \begin{vmatrix} 1 & b & d & e \\ 1 & c & e & a \\ 1 & e & b & c \\ 1 & a & c & d \end{vmatrix} + \begin{vmatrix} 1 & c & d & e \\ 1 & e & a & b \\ 1 & a & b & c \\ 1 & b & c & d \end{vmatrix} = 0,$$

where a, b, c, d, e are the elements of a circulant of order five. He obtained it as the coefficient of the first power of x in Lagrange's equations, which power (as well as all the odd powers) was proven not to exist, and next enunciates the following general theorem: *If the elements of the first column of any odd-ordered circulant, axisymmetric with respect to the principal diagonal, be replaced by units, the sum of the complementary minors of the elements in the places* (2, 2), (3, 3), $\cdots$, (n, n) *vanishes.*

He next points out that in the case $n = 7$ we may substitute for

$$[2, 2]_1 + [3, 3]_1 + [4, 4]_1 + [5, 5]_1 + [6, 6]_1 + [7, 7]_1 = 0,$$

the two relations

$$[2, 2]_1 + [3, 3]_1 + [5, 5]_1 = 0,$$

$$[4, 4]_1 + [6, 6]_1 + [7, 7]_1 = 0,$$

where $[p, q]_r$ denotes the complementary minor of the element in the pth row and qth column after the rth column of the circulant has been replaced by units.

[1] For the purposes of this paper the following definition will be assumed: If each row of a determinant may be derived from the preceding row by passing the first element over all the others to the last place, the determinant is called a circulant.

[2] "Lagrange's determinantal equation in the case of a circulant," *Messenger of Mathematics*, New Series, vol. 41, March, 1912.

The object of this note is to investigate these aggregates a little more closely and from a somewhat different viewpoint, and to determine those that are fundamental.

By equating the coefficients of odd powers of x to zero we would get various vanishing aggregates connecting minors of even order but these aggregates would not be fundamental.

Starting with the determinant

$$\begin{vmatrix} b & c+a & d+e & 1 \\ c & d+b & e+a & 1 \\ d & e+c & a+b & 1 \\ e & a+d & b+c & 1 \end{vmatrix},$$

which obviously vanishes since two columns may be made identical, we have

$$\begin{vmatrix} b & c & d & 1 \\ c & d & e & 1 \\ d & e & a & 1 \\ e & a & b & 1 \end{vmatrix} + \begin{vmatrix} b & c & e & 1 \\ c & d & a & 1 \\ d & e & b & 1 \\ e & a & c & 1 \end{vmatrix} + \begin{vmatrix} b & a & d & 1 \\ c & b & e & 1 \\ d & c & a & 1 \\ e & d & b & 1 \end{vmatrix} + \begin{vmatrix} b & a & e & 1 \\ c & b & a & 1 \\ d & c & b & 1 \\ e & d & c & 1 \end{vmatrix} = 0,$$

or

$$[1, 4]_5 - [1, 5]_3 + [1, 2]_4 - [1, 3]_2 = 0. \tag{1}$$

From the properties of circulants of odd order we have

1. When $1 + r$ is even

$$[1, r]_s = \left[\frac{1+r}{2}, \frac{1+r}{2}\right]_{s-[(r-1)/2]} \text{ if } s > \frac{r-1}{2},$$

$$= \left[\frac{1+r}{2}, \frac{1+r}{2}\right]_{n+s-[(r-1)/2]} \text{ if } s \leqq \frac{r-1}{2}.$$

2. When $1 + r$ is odd

$$[1, r]_s = -\left[\frac{n+r+1}{2}, \frac{n+r+1}{2}\right]_{s-[(n+r-1)/2]} \text{ if } s > \frac{n+r-1}{2},$$

$$= -\left[\frac{n+r+1}{2}, \frac{n+r+1}{2}\right]_{s+[(n-r+1)/2]} \text{ if } s \leqq \frac{n+r-1}{2}.$$

We also have

1. When $r + s$ is even

$$[r, r]_s = -\left[\frac{r+s}{2}, \frac{r+s}{2}\right]_{n+[(3r-s)/2]} \text{ if } 3r \leqq s,$$

$$= -\left[\frac{r+s}{2}, \frac{r+s}{2}\right]_{(3r-s)/2} \text{ if } 3r > s;$$

2. When $r + s$ is odd

$$[r, r]_s = -\left[\frac{n+r+s}{2}, \frac{n+r+s}{2}\right]_{n+[(3r-n-s)/2]} \text{ if } 3r \leqq s+n,$$

$$= -\left[\frac{n+r+s}{2}, \frac{n+r+s}{2}\right]_{(3r-n-s)/2} \text{ if } 3r > s+n.$$

Making use of these relations (1) takes the form

$$[2, 2]_1 + [3, 3]_1 + [4, 4]_1 + [5, 5]_1 = 0,$$

which is as given by Muir.

If we had started with

$$\begin{vmatrix} b & c+e+a & d & 1 \\ c & d+a+b & e & 1 \\ d & e+b+c & a & 1 \\ e & a+c+d & b & 1 \end{vmatrix},$$

which is also equal to zero, we would have

$$[1, 4]_5 - [1, 5]_2 + [1, 2]_4 = 0. \tag{2}$$

Similarly

$$[1, 3]_5 + [1, 5]_2 - [1, 2]_3 = 0. \tag{3}$$

From (2) and (3) we get

$$[1, 3]_5 - [1, 2]_3 + [1, 4]_5 + [1, 2]_4 = 0,$$

or

$$[3, 3]_1 + [2, 2]_1 + [5, 5]_1 + [4, 4]_1 = 0,$$

which is (1).

Again

$$[1, 1]_2 + [3, 3]_2 + [4, 4]_2 + [5, 5]_2 = 0;$$

and since $[5, 5]_1 = -[3, 3]_2$, we have

$$[1, 1]_2 + [4, 4]_2 + [5, 5]_2 + [2, 2]_1 + [3, 3]_1 + [4, 4]_1 = 0. \tag{4}$$

The general law for (2) is, in the case of circulants of odd order,

$$(-1)^{r-1}[1, r]_s + (-1)^{s-1}[1, s]_t + (-1)^{t-1}[1, t]_r = 0. \tag{5}$$

The general law for (1) is

$$(-1)^{r-1}[1, r]_s + (-1)^{s-1}[1, s]_t + (-1)^{t-1}[1, t]_u + (-1)^{u-1}[1, u]_r = 0, \tag{6}$$

but, as has been seen, (6) is made up of two of (5).

The method here used will give vanishing aggregates of minors of any order. Thus from

$$\begin{vmatrix} 1 & a+b+c+d & c \\ 1 & b+c+d+e & a \\ 1 & c+d+e+a & b \end{vmatrix} = 0,$$

where as before, a, b, c, d, e are the elements of a circulant of order five, we have

$$\begin{vmatrix} 1 & a & e \\ 1 & b & a \\ 1 & c & b \end{vmatrix} + \begin{vmatrix} 1 & b & e \\ 1 & c & a \\ 1 & d & b \end{vmatrix} + \begin{vmatrix} 1 & c & e \\ 1 & d & a \\ 1 & e & b \end{vmatrix} + \begin{vmatrix} 1 & d & e \\ 1 & e & a \\ 1 & a & b \end{vmatrix} = 0,$$

or

$$-\begin{bmatrix} 45 \\ 23 \end{bmatrix}_4 + \begin{bmatrix} 45 \\ 34 \end{bmatrix}_1 - \begin{bmatrix} 45 \\ 41 \end{bmatrix}_2 + \begin{bmatrix} 45 \\ 12 \end{bmatrix}_3 = 0,$$

or

$$-\begin{bmatrix} 12 \\ 12 \end{bmatrix}_5 + \begin{bmatrix} 12 \\ 12 \end{bmatrix}_4 - \begin{bmatrix} 15 \\ 15 \end{bmatrix}_3 + \begin{bmatrix} 15 \\ 15 \end{bmatrix}_2 = 0,$$

where $\begin{bmatrix} r & s \\ t & u \end{bmatrix}_v$, is the complementary of the minor of the elements in the intersection of the rth and sth rows, and the tth and uth columns, after the elements in the vth column of the circulant are replaced by units.

Using this same method we may also obtain vanishing aggregates of minors of circulants of even order. Thus for $n = 4$ we have

$$\begin{vmatrix} 1 & a+b+c & d \\ 1 & b+c+d & a \\ 1 & c+d+a & b \end{vmatrix} = 0,$$

and

$$\begin{vmatrix} 1 & a & d \\ 1 & b & a \\ 1 & c & b \end{vmatrix} + \begin{vmatrix} 1 & b & d \\ 1 & c & a \\ 1 & d & b \end{vmatrix} + \begin{vmatrix} 1 & c & d \\ 1 & d & a \\ 1 & a & b \end{vmatrix} = 0,$$

or

$$-[1, 1]_2 + [1, 2]_4 + [1, 4]_1 = 0,$$

which is in accord with (5).

Again

$$\begin{vmatrix} 1 & a+c & b+d \\ 1 & b+d & c+a \\ 1 & c+a & d+b \end{vmatrix} = 0,$$

and

$$[1, 2]_3 - [1, 3]_4 + [1, 4]_1 - [1, 1]_2 = 0,$$

which is in accord with (6). It is otherwise evident that we may remove from (5) and (6) the restriction that n must be odd, and have them true for any order of circulant.

November, 1918.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

We present this month only one discussion, by Professor Florence P. Lewis, dealing with Euclid's parallel postulate. Professor Lewis sketches the history of the controversy over this postulate, and indicates its bearing on the problems of teaching. The subject is of great interest and importance. While it would be rash to assert that a knowledge of the history of non-Euclidean geometry and an acquaintance with the modern views regarding the nature of a postulate are indispensable prerequisites to successful teaching of elementary geometry, nevertheless it can not be denied that such equipment must notably enrich and vivify a teacher's own appreciation of the subject and thus add materially to the effectiveness of teaching.

The subject is an extensive one, not readily amenable to adequate treatment in a short article. It is unlikely that any two persons, in giving a short account of the *history* of the parallel postulate and the development of non-Euclidean geometry, would make exactly the same selection of names to be mentioned or the same interpretation of historical facts. Still less likely would be the agreement of independent writers on the *pedagogic* aspects of the question. Thus our readers will probably occasionally disagree with parts of Professor Lewis's treatment or dissent from some of her conclusions. But it is believed that the article as a whole represents a consistent and just account, which should be of value to many readers, especially those concerned with the teaching of geometry.

Professor Lewis's contention that we should cease to fear redundancy in our list of assumptions seems irrefutable. A course in geometry for adolescents should not be planned in the same way as a course for graduate students in the university. Why "prove" to a high school student that circles with equal radii are congruent? To the poor student the proof brings no added conviction; while the good student wonders why, if this proof is logically necessary, it is not even more essential to show that no arc of a circle is a segment of a straight line. Both statements would be welcome as assumptions.

Perhaps it is not easy to draw the line between what is to be assumed and what is to be proved. Probably every child would accept as assumptions the propositions on congruence of triangles; probably almost none would be satisfied to accept the angle sum of a triangle without proof. It may require some discrimination to decide just which theorems arouse in the student that *demand for an answer* which is the kernel of all successful teaching; but there can be little doubt that such decision should form the basis of our future treatment of elementary geometry.

History of the Parallel Postulate.[1]

By Florence P. Lewis, Goucher College.

Like the famous problems of construction, Euclid's postulate concerning parallels is a thought that links the ages. Its history is a long story with dramatic climax and far-reaching influence on modern mathematical and general scientific thought. I wish to recall briefly the salient features of the story, and to state what seem to me its suggestions in regard to the teaching of elementary geometry.

Euclid's fifth postulate (called also the eleventh or twelfth axiom) states: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which are the angles less than two right angles." The earliest commentators found fault with this statement as being not self-evident. Concerning the meaning of *axiom*, Aristotle says: "That which it is necessary for anyone to hold who is to learn anything at all is an axioms;" and "It is ignorance alone that could lead anyone to try to prove the axiom." Without going into the difficult question of the precise distinction to the Greek mind between axiom and postulate, we may take it that the character of being indisputable pertained to each. Postulates stating that a straight line joining any two points can be drawn, that a circle can be drawn with given center and radius, or that all right angles are equal, were accepted, while the postulate of parallels was scrutinized and admitted at best with reluctance.

Proclus, writing in the fifth century A. D., gives some of the reasons for this attitude, and we may surmise others. The postulate makes a positive statement about a region beyond the reach of possible observation or geometrical intuition. Proclus insinuates that those who "suppose they have ground for instantaneous belief" are "yielding to mere plausible imaginings"; the conclusion is "plausible but not necessary."[2] The converse is proved in Proposition 27, book I of Euclid's *Elements*, and there seems to be no reason why this proposition should be more or less self-evident than its converse. The fact that the two lines continually approach each other[3] was not a convincing argument to the Greek geometer who was acquainted with the relation of the hyperbola to its asymptote. The form of statement of the postulate is long and awkward compared with that of the others, and its obviousness thereby lessened. There is evidence that Euclid himself endeavored to prove the statement before putting it down as a postulate; for in some manuscripts it appears not with the others but only just before Proposition 29, where it is indispensable to the proof. If the order is significant, it indicates that the author did not at first intend to include this among the postulates, and that he finally did so only when he found that he could neither prove it nor proceed without it.

[1] Read before the Association of Teachers of Mathematics in New England, May 3, 1919.

[2] Cf. Heath's Euclid, Vol. I, and Bonola's *Non-Euclidean Geometry*, Chicago, Open Court, 1912, to which reference is made throughout this paper.

[3] Even the meaning of this phrase requires further elucidation.

Most of the early geometers appear to have attacked the problem. Proclus quotes and criticizes several proofs, and gives one of his own. He instances one writer who even attempted to prove the falsity of the statement, the argument being similar to those used in Zeno's paradoxes. The common opinion, however, seems to have been that the postulate stated a truth, but that it ought to be proved. Euclid had proved two sides of a triangle greater than the third, which is far more obvious than this. If the statement was true it should be proved in order to convince the doubters; if false, it should be removed. In no case should it be retained among the fundamental presuppositions. Sir Henry Savile (1621) and the Italian Saccheri (1733) refer to it as a blot or blemish on a work that is otherwise perfect, and this expresses the common attitude of mathematicians until the first quarter of the nineteenth century.

Early attempts at proof usually took the form of a change in the definition of parallels, or the substitution, conscious or unconscious, of a new assumption. Neither of these methods resulted in satisfaction to any but their inventors; for the definitions usually concealed an assumption, and the new postulates were no more obvious than the old. Posidonius, quoted by Proclus, defines parallels as lines everywhere equidistant. This begs the question; surely such parallels do not meet, but may there not be in the same plane other lines, not equidistant,which also do not meet? The definition involves also the assumption, that the locus of points in a plane at a given distance from a straight line is a straight line, and this was not self-evident.[1] Ptolemy says that two lines on one side of a transversal are *no more parallel* than their extensions on the other side; hence if the two angles on one side are together less than two right angles, so also are the two angles on the other side, which is impossible since the sum of the four angles is four right angles. This is another way of saying that through a point but one parallel to a given line can be drawn, which is exactly Euclid's postulate. Proclus himself assumes (with some concealment) that if a line cuts one of two parallels it cuts the other, which is again postulate 5. Even as late as the close of the eighteenth century we find this argument advanced by one Thibault, and attributed also to Playfair: Let a line segment with one end A at a vertex of a triangle be rotated through the exterior angle. Translate it along the side until A comes to the next vertex and repeat the process. We finally arrive at the original position and must therefore have rotated through 360°. Hence the sum of the interior angles of a triangle is 180°; and, since Legendre had satisfactorily proved that this proposition entails Euclid's postulate of parallels, the latter is at last demonstrated. The fact that the same process could equally well be carried out with a spherical triangle, in which the angle-sum is not 180°, might have given him pause. The assumption that translation and rotation are independent operations is in fact equivalent to Euclid's postulate.

[1] It should be noted that even the meaning of the criterion suggests several questions of logic. If two lines are so placed that perpendiculars to one of them from points on the other are equal, will the same statement hold when the rôles of the two lines are reversed? Will a perpendicular to one of two non-intersecting lines necessarily be perpendicular to the other? Of course the answers to these questions are closely bound up with the very postulate under discussion.—EDITOR.

Heath gives (l. c.) a long and instructive list of these substitutes. In the course of centuries the minds of those interested became clear on one point: they did not wish merely to know whether it was possible to substitute some other assumption for Euclid's, though this question has its interest; they wished to know primarily whether exactly his form of the postulate was logically deducible from his other postulates and established theorems. To change the postulate was merely to re-state the problem.

After certain Arabs and Persians had had their say in their day, the curtain rises on the Italian Renaissance of the sixteenth century, where the problem was attacked with great vigor. French and British assailants were not lacking. The first modern work devoted entirely to the subject was published by Cataldi in 1603. When the eighteenth century took up the unfinished business of proving the parallel postulate, we find most of the giants of those days attacking the enemy of geometers with an even keener sense that without victory there could be no peace. Yet d'Alembert toward the close of the century could still refer to the state of the theory of parallels as "the scandal of elementary geometry." Klügel in 1763 examined thirty demonstrations of the postulate. He was perhaps the first to express doubt of its demonstrability. Lagrange, according to De Morgan, in about 1800, when in the act of presenting to the French Academy a prepared memoir on parallels, interrupted his reading with the exclamation, "Il faut que j'y songe encore," and withdrew his manuscript.

While the results of these investigations were on the whole negative, certain positive and valuable results were nevertheless obtained. The relation between the parallel postulate and the angle sum of a triangle was clearly brought out. Legendre proved that if in a single triangle the angle sum is two right angles, the postulate holds. Other equivalents are of interest. John Wallis and Laplace wished to assume: There exists a figure of arbitrary size similar to any given figure. Gauss could proceed rigorously provided he could prove the existence of a rectilinear triangle whose area is greater than any previously assigned area. W. Bolyai could have succeeded with the assumption that a circle can be passed through any three points not in a straight line. It must be borne in mind, moreover, that few mathematical questions have served so well as whetstones on which to sharpen the critical powers of mankind.

The work of the Italian priest Saccheri deserves notice because his method is that which finally brought the discussion to a close. Though published in 1733 his results did not become well known until after 1880, and therefore had little influence on other investigations. Legendre's *Réflexions*, published a hundred years later, covered much of the same ground without advancing quite so far. The title of Saccheri's work is *Euclides ab omni Naevo Vindicatus*, Euclid Vindicated of every Flaw. His plan was to prove the postulate by assuming its contradictory and showing that an inconsistency followed. He succeeded in proving that, according as in one triangle the angle sum is greater than, equal to, or less than two right angles, the same holds in every triangle, and that accordingly Euclid's postulate or one of its contradictories will hold.

He makes three hypotheses which were recognized later to correspond to the elliptic, Euclidean and hyperbolic geometries. But at the end of his work, in order to exhibit a contradiction when Euclid's postulate is denied, he is forced to make use of a somewhat vague and unacceptable assumption about "the nature of a straight line."

Gauss's activity in connection with the parallel postulate is of especial interest because of its psychologic aspect. It is difficult for us to picture a mathematician hesitating to publish a discovery for fear of the outcry that its publication might produce—perhaps not many would be displeased to awaken an echo; yet this is believed by some to have been the attitude of Gauss. Though he was keenly interested and thought deeply on the subject of parallels, he published nothing; he feared, as he said, "the clamor of the Bœotians." When forced to write a letter on the subject, he begs his correspondent to keep silence as to the information imparted. In 1831 he writes in a letter: "In the last few weeks I have begun to put down a few of my Meditations [on parallels] which are already to some extent forty years old. These I had never put in writing, so that I have been compelled three or four times to go over the whole matter afresh in my head. Also I wished that it should not perish with me." It is only when we call to mind the unrivalled place of honor held by Euclidean geometry among branches of human knowledge—a respect no doubt enhanced by the prominence given it in Kant's *Critique of Pure Reason*—that we realize the uncomfortable position of one who even appeared to attack its validity. Gauss's meditations were leading him through tedious and painstaking labors to the conclusion that Euclid's fifth postulate was not deducible from his other postulates. The minds of those not conversant with the intricacies of the problem might easily rush to the conclusion that Euclid's geometry was therefore untrue, and feel the whole structure of human learning crashing about their ears.

Between 1820 and 1830, the conclusion toward which Gauss tended was finally made sure by the invention of the hyperbolic non-Euclidean geometry by Lobachevsky and Johann Bolyai, working simultaneously and independently. The question, Is Euclid's fifth postulate logically deducible from his other postulates? is answered by showing that the denial of this postulate while all the others are retained leads to a geometry as consistent as Euclid's own. The method, we recall, was that used by Saccheri, whose intellectual conservatism alone prevented his reaching the same result. The famous postulate is only one of three mutually exclusive hypotheses which are logically on the same footing. Thus was Euclid "vindicated" in an unexpected manner. Knowingly or not, the wise Greek had stated the case correctly, and only his followers had been at fault in their efforts for improvement. To quote Heath: "We cannot but admire the genius of the man who concluded that such an hypothesis, which he found necessary to the validity of his whole system, was really indemonstrable."

Thus in some sense the problem of the parallel postulate was laid to rest, but its spirit marches on. If the fifth postulate could without logical error be replaced by its contradictory, could the other postulates be similarly treated? What is

the nature of a postulate or axiom? What requirements should a satisfactory system of axioms fulfill? Are we sure that accepted proofs will bear as keen scrutiny as that to which proofs of the postulate have been subjected? The facing of these questions has brought us to the modern critical study of the foundations of geometry. It has been realized that if geometry is to continue to enjoy its reputation for logical perfection, it should at least try to deserve it. The edge of criticism, sharpened on the parallel postulate, is turned against the whole structure. Out of this movement has grown the critical examination of the foundations of algebra, of projective geometry, of mechanics, of logic itself; and the end is not in sight.

One obvious result of this critical study is that geometrical axioms are not necessary truths, but merely presuppositions: they are the hypotheses on which the whole body of theorems rests. It is essential that a system of axioms should be consistent with each other, and desirable that they be non-redundant, and complete. No one has found Euclid's system inconsistent,[1] and redundancy would be a crime against elegance rather than against logic. But on the score of completeness Euclid is far from giving satisfaction. He not infrequently states conclusions which could be arrived at only by looking at a figure, *i.e.*, by space intuition; but we are all familiar with cases where space intuition misleads (for example in the fallacious proof that all triangles are isosceles), and if we accept it as a guide how can we be sure that our intuitions will always agree? In constructing an equilateral triangle Euclid says, "From point C where the two circles meet, draw $\cdots$." Perhaps they do meet—but not on the basis of anything previously stated. In dropping a perpendicular from a point to a line, he says a certain circle will meet the line twice. Why should it not cut thrice or not at all? In another proof he says that a certain line will lie within a certain angle. I *see* that it does, but I do not see it proved. We are told in the midst of a proof to bisect an angle of a triangle and produce the bisector to meet the opposite side. How do we know it will meet? Because it is not a parallel. And probably it is not parallel because it is inside the triangle. How do we know it is inside; or, being inside, that it must get outside? When have these terms been defined? You may answer: It is not necessary to define them because everyone with common sense knows inside from outside without being told. "Who is so dull as not to perceive $\cdots$?" says Simson, one of Euclid's apologists. This may be granted. But it must be pointed out that common sense knows that two straight lines cannot enclose space, yet this is given prominence as an axiom; or that a straight line is the shortest distance between two points, yet this is proved as a proposition. To state in words what distinguishes the inside from the outside of a polygon is not easy. The word "between" is likewise difficult of definition. But the modern geometer imbued with the critical spirit feels it necessary to define such terms, and what is more, he finds a way of doing it.

[1] It is in fact possible to show that the system is consistent, provided we agree to accept the axioms of arithmetic as consistent. While this is only a transformation of the problem, it is logically important to recognize the possibility of such a transformation.—EDITOR.

Hilbert's *Grundlagen der Geometrie*, published in 1899, is a classic product of this movement. It presupposes no space concepts at all, but only such general logical terms as "corresponds to," "associated with," "determined by." Contrary to tradition, it does not begin by defining terms. The first sentence is: "I think of three systems of things which I call points, lines and planes." Note the unadorned simplicity of the concept *things*. The axioms serve as definitions. They state, in non-spatial terms, relations between these "things"; that is to say, the points, lines and planes are such things as have such and such relations. "That is all ye know on earth, and all ye need to know." Twenty-one axioms are found necessary, as against Euclid's meager five. The whole work could be read and comprehended by a being with no space intuitions whatever. We could substitute the names of colors or sounds for points, lines and planes, and get on equally well. The ideal of making a thing "so plain that a blind man could see it" is literally realized. And the age-old ideal of a body of proved propositions, close-knit together by unassailable logic, is immeasurably nearer realization.

Although to the best of my knowledge no one has yet had the hardihood to invite a child of fourteen to consider "three systems of things," [1] the modern critical movement is not without bearing on problems of teaching. I wish, with proper humility, to put forward a few ideas on this subject.

If it is true that our traditional formal geometry, taken directly or indirectly from Euclid, is not the logically perfect thing we had imagined, and if its modern perfected descendant is so abstract that not even the most rationalistic of us would venture to force it on beginners, why not acknowledge these facts and bravely face anew the question of how we can best make the study of elementary geometry serve its proclaimed purpose of training the mind? I would suggest two lines along which progress might be made. First, by sacrificing the ideal of non-redundancy in our underlying assumptions we could save time and stimulate interest by arriving more quickly at propositions whose truth is not immediately evident and which could be presented as subjects for investigation. Must we, because Euclid did it, prove that the base angles of an isosceles triangle are equal? A child that has cut the triangle out of paper and folded it over knows as much as any proof can teach him. If to treat the proposition in this way is repugnant to the teacher's logical conscience, let him privately label it "axiom" or "postulate," and proceed, even though this proposition could have been proved. *The place to begin producing arguments is the place where the truth of the proposition is even momentarily in doubt.* One statement which presents itself with a question mark and is found after investigation to be true or to be false is worth ten obviously true statements proved with all the paraphernalia of hypothesis, conclusion, step one and step two, with references. The only apparent reason for proving in the traditional way the theorems on the isosceles triangle and congruent triangles is in order to familiarize the student with the above-mentioned paraphernalia. This brings me to my second suggestion.

Formal geometry has been looked upon as a complete and perfect thing to

[1] It is possible that Halsted's *Rational Geometry* has a somewhat similar purpose.—EDITOR.

which the learner can with profit play the sedulous ape. Yet I sometimes think that by emphasizing too early the traditional form of presentation of geometrical argument, and paying too little attention to the psychology of the learner, we may have corrupted some very good minds. "I wish to prove . . .," says the student; meaning, I wish to prove something stated and accepted as true in advance of argument. Should we not prefer to have our students say, "I wish to examine . . ., to understand, to find out whether . . ., to discover a relation between . . ., to invent a means of doing . . ."? What better slogan could prejudice desire than "I wish to prove"? The conscienceless way in which college debaters collect and enumerate arguments regardless of the issues involved is another aspect of the same evil. A student said not long ago, "The study of mathematics would be good fun if we did not have to learn proofs." It had never been brought home to her that mathematical reasoning is not a thing to be acquired, like a knowledge of Latin verbs, but a thing to be participated in like any other form of exercise. Another said, "I cannot apply my geometry because all we did in school was to learn the proofs and pass the examinations."

In the midst of a proof the student hesitates and says, "I am sure this is the next step, but I cannot recall the reason for it." The step and its reason would occur simultaneously to a mind that had faced the proposition as a problem and thought it through. I should like to see in every text an occasional page of exercises to prove or disprove. And if formal proofs must be printed in full, by all means let some of the proofs be wrong.

When the student has thus halted with one foot in the air in this progress from step to step down the printed page, on what does the ability to proceed depend? On the ability to quote something: to quote, usually, a single statement—compact, authoritative, triumphantly produced. Surely it is bad training that leads the mind to *expect* to find support for its surmises in a form so simple; and the temptation to substitute ability to quote in place of the labor of finding out the truth may be a real danger. What wonder if the mind so trained quotes Washington's Farewell Address or the Monroe Doctrine and feels that its work is done?

I do not mean that formal proof should never be given. It has its place as an exercise in literary composition; for it deals with the form in which thought is expressed. We should, however, take every possible precaution to see that the thought is first there to be expressed, lest the form be mistaken for the substance. Just how this is to be brought about I am not prepared to discuss, although I suspect that drawing and measurement in the early stages of study, problems of construction and investigation, and the total absence of complete proofs from the printed page, would help. I wish merely to state my belief that only in so far as we succeed in these aims shall we succeed in making geometry really train the mind.

> It can be done, said the butcher, I *think;*
> It must be done, I am sure.

One point further. Perhaps we are a little too modest about the importance

of having our students retain something of the subject-matter of the courses we teach. Evidently it is here that memory, based on understanding, should rightly be used. I sometimes think we might in some way collectively take out insurance against a student's arriving at the junior year in college in the belief that two triangles are similar whenever they have a side in common.

RECENT PUBLICATIONS.

REVIEW.

Introductory Mathematical Analysis. By W. PAUL WEBBER and LOUIS C. PLANT. New York, Wiley, 1919. 13 + 304 pages. Price $2.00.

This combination book for freshmen covers plane trigonometry and topics from advanced algebra, analytic geometry, and the calculus, which are treated in the order named.

The first seven chapters (70 pages) may be said to form an introduction containing a review of algebra and geometry, and such general topics as computations, including logarithms and the slide rule, rectangular coördinates with graphical representations of statistics and equations, classes of numbers, variables, functions, and limits. Interpolation in tables involving two independent variables is given, and simple problems to determine the form of an equation representing a given empirical table are solved. In a later chapter some problems of the latter sort are solved by using logarithmic and semi-logarithmic paper.

Chapter VIII (50 pages) gives the usual course in plane trigonometry in condensed form. In defining the six functions a tabular form is used in which a row is given to each function and a column to an angle in each of the four quadrants, a point $P_1(x_1,y_1)$ being chosen on the terminal line of an angle α_1 in the first quadrant, a point $P_2(x_2, y_2)$ on the terminal line of an angle α_2 in the second quadrant, etc. The table includes, in parentheses, the sign of each function for each quadrant. A radian "is defined by the equation Arc $AB = r \cdot \theta$, where r is the radius of the arc AB and θ is the angle in radians." Portions of a traverse table and of a table of haversines, and some problems in indirect fire, appear among the applications.

The chapter on the circular functions is followed by one on polar coördinates, complex numbers, and vectors, which connects trigonometry with algebra. Vector and scalar products are given considerable attention, with application to equilibrium of particles and rigid bodies. A short chapter on equations gives a few of the theorems from the theory of equations, but the opportunity to tie it to the preceding chapter by the theorem that complex roots occur in conjugate pairs was not improved. The reviewer holds no brief for this theorem, especially as he thinks other material more suitable for and valuable to freshmen than complex numbers, but the omission is noticeable in a text which considers the derivative of u^n for a complex value of n, which treats series with complex terms, and which expresses the sine and cosine in the well-known exponential forms in order to

obtain their derivatives. Real roots of equations of higher than the second degree are approximated by synthetic division without shifting the y-axis.

The chapters on analytic geometry (41 pages) are devoted chiefly to the straight line and conic sections. These are treated very compactly, but with the completeness of many texts. Space is saved in the chapter on conics by leaving much of the theory to be worked out as exercises. For example, the equation of an ellipse is derived in the form

$$(1 - e^2)x^2 - 2epx + y^2 = 0,$$

and the following exercises given.

"2. Move the origin to the point $(ep/(1 - e^2), 0)$ to remove the term in the first power of x."

"3. Call $ep/(1 - e^2) = a$ and $(1 - e^2)a^2 = b^2$ in the equation obtained in example (2) and show that the equation reduces to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1."$$

Nearly a third of the book (88 pages) is given to the calculus. The ϵ, δ form of proof is used for the preliminary theorems on limits, the derivative introduced in a purely formal way, and the theorems necessary for differentiating algebraic functions, implicit as well as explicit, are obtained before it comes to light that the derivative has any concrete significance, either as the slope of a tangent line or as a rate. Series receive an extended treatment. Maclaurin's series is obtained by differentiating an assumed expansion and determining the coefficients. The function e^x is defined as a series, and the expansion of $e^{\Delta u}$ is the basis for obtaining the derivative of e^u. Integration is considered as the inverse of differentiation, and also as the limit of a sum. The applications of the calculus most emphasized are tangent lines, maxima and minima, velocity and acceleration, plane areas, volume of a solid of revolution, work, and centroids.

Much of the material is presented in a very compact form, so that, as stated in the preface, "The teacher will find an opportunity for originality in developing the text and at times a necessity for more details." The authors' style is sometimes abrupt, as in the sentence "Let student solve and verify," and a few sentences are open to more serious criticisms. For example: "Let x be a variable and a some constant. Then if $a - x$ assumes its sequence of values in order, there comes a stage, such that, for all subsequent values of x, the numerical value of $a - x$ becomes and remains less than any assigned small value ϵ, then a is called the **limit** of x" (page 64). "If, at a given instant of time, the speed of a body becomes uniform, then the distance Δs, passed over in the time Δt, is the instantaneous speed at the given instant" (page 214). The treatment of instantaneous speed which follows the latter quotation is unsatisfactory in that it confuses two different meanings of Δs.

ARTHUR SULLIVAN GALE.

BRIER HILL, N. Y.,
August, 1919.

NOTES.

"In 1880 the Cambridge University Press began the republication in collected form of Stokes's *Mathematical and Physical Papers*. In this publication he introduced for the first time the *solidus* notation for division, originally introduced by De Morgan in his article on the Calculus of Functions in the *Encyclopaedia Metropolitana*. If a fraction like $\frac{a}{b}$ or a differential coefficient such as $\frac{dy}{dx}$, is mentioned in the text, the printing of such expressions requires a good deal of "justification" on the part of the compositor. To avoid this expense and the loss of space Stokes introduced the linear notation a/b and dy/dx." [A. Macfarlane's lecture on "Sir George Gabriel Stokes" in *Lectures on Ten British Physicists of the Nineteenth Century* (New York, 1919), pp. 100–101.]

The Mathematical Association of Japan for Secondary Education issued the first number of Volume 1 of its *Journal* under date of April, 1919. It is published by M. Kaba, Tokyo, as chief editor, assisted by M. Kuroda, M. Watanabe, S. Nagao and H. Furnkawa. While all papers in this issue were printed in Japanese, the announcement is made that contributions in English are accepted. The contents of the initial issue are as follows:

Treatise

Presidential Address by T. Hayashi.

Miscellanies

On the graph of $\log x$ by M. Kaba; list of market prices and some technical terms in business circles by M. Kaba; on some syllabi of mathematics in the European and American secondary schools by M. Kuroda.

Problems for Consideration

On entrance examination by M. Kaba; on the equipment for teaching of mathematics by H. Furnkawa; on the teaching of arithmetic in the middle school course by M. Kuniyeda; on the teaching of graphs in the middle school course by M. Kuroda.

Mathematical Problems by M. Watanabe; Book Reviews; Questions; Miscellaneous Reports; Social Column; Special Reports.

ARTICLES IN CURRENT PERIODICALS.

BULLETIN DES SCIENCES MATHEMATIQUES, volume 54, March, 1919: Review by R. Garnier of E. Turrière's *Sur le calcul des objectifs astronomiques de Fraunhofer* (1917), 49–50; "Sur un théorème relatif a l'extension du théorème de Rolle aux fonctions de plusieurs variables" by E. Gau, 50–51; "Sur le calcul des perturbations" by H. Vergne, 51–72; Revue des publications, 17–24.—April: Review by A. Boulanger of P. Duhem's *Etudes sur Léonard de Vinci*. Troisième série (Paris, 1913), 73–77; "Sur le calcul des perturbations" (suite et fin) by H. Vergne, 78–79; "La série $\frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \frac{1}{29} + \frac{1}{31} + \frac{1}{41} + \frac{1}{43} + \frac{1}{59} + \frac{1}{61} + \cdots$ où les dénominateurs sont 'nobres premiers jumeaux' est convergente ou finie" by V. Brun, 100–104 (à suivre).

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 26, no. 1, October, 1919: "In memory of Gabriel Marcus Green" by E. J. Wilczynski, 1–13 [Quotation: "In the six short years of his mathematical career, from 1913–1919, he enriched geometry with so many new ideas and important results as would suffice to excite our admiration if they had been spread over all of a normal life time. In his death we have suffered a heavy loss, but his life and work will continue to be, for many of us, an everlasting source of strength and inspiration"]; "Reduction of the elliptic element to the Weierstrass form" by F. H. Safford, 13–16; "A note on 'continuous mathematical induction'" by Y. R. Chao, 17–18; "On the number of representations of $2n$ as a sum of $2r$ squares" by E. T. Bell, 19–25; "Some functional equations in the theory of relativity" by A. C. Lunn, 26–34; "Formulas for constructing abridged mortality tables for decennial ages" by C. H. Forsyth, 34–38; Review by F. W. Owens of Dowling's *Projective Geometry* (New York, 1917), 39–40; Review by C. F. Craig of McClenon's *Introduction to the Elementary Functions* (Boston, 1918), 40–41; "Notes" and "New Publications," 41–48.

BULLETIN OF THE FIRST DISTRICT NORMAL SCHOOL, Kirksville, Mo., volume 15, no. 9, September, 1915; Mathematics Series no. 1: "The use of the graph in geography teaching" by B. Cosby, 7–8; "A plan for the study of stocks and bonds" by W. H. Zeigel, 9–14; "The history of mathematics as an incentive to mathematical study" by G. H. Jamison, 15–20; "The value of a life setting to algebra problems" by B. Cosby, 21–25; "Analytic geometry in our high schools" by W. H. Zeigel, 26–32; "Sets of orthogonal functions and their oscillation properties" by C. A. Epperson, 33–38; "A teacher's library" by G. H. Jamison, 38–39; "Henry Ward Beecher and mathematics," 40—Volume 16, no. 10, October, 1916; Mathematics Series no. 2: "Building and loan associations explained" by W. H. Zeigel, 5–10; "An experiment in teaching algebra" by C. A. Epperson, 11–13; "Some suggestions and observations on the teaching of high-school mathematics" by G. H. Jamison, 14–20; "The purpose and content of high-school arithmetic" by B. Cosby, 21–30; "The cube root of a binomial surd" by W. H. Zeigel, 31–38—Volume 18, no. 12, December, 1918 (published August, 1919). Mathematics Series No. 3[1]: "A geometrical problem —proof and application" by W. H. Zeigel, 5–10; "Keeping abreast of the times" by G. H. Jamison, 11–13; "Investments" by Byron Cosby, 14–20; "A suggested problem for classes in analytic geometry and surveying" by C. A. Epperson, 21–23; "The rôle of assumptions and definitions in high school mathematics" by G. H. Jamison, 24–30.

JOURNAL OF THE UNITED STATES ARTILLERY, Volume 50., June 1919: "A method of computing differential corrections for a trajectory" by G. A. Bliss, 455–460 [reprinted from corrected proof in volume 51 (October), 445–449]—Volume 51, August: "Equations of differential variations in exterior ballistics" by W. E. Milne, 154–159—September: "The new ballistics" by R. S. Hoar, 285–295; "The use of adjoint systems in the problem of differential corrections for trajectories" by G. A. Bliss, 296–311; "Effect of the earth's rotation upon the point of fall" by P. Field, 328–329—October: "Rotating bands" by O. Veblen and P. L. Alger, 355–390.

Nature, volume 104, September 25, 1919: "Mathematics at the University of Stressburg" by H. B. Heywood, 74; The "algebraic cube," 79 [model in eight pieces illustrating the formula $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. "The blocks are supplied in a neat cubical box, 10 cm. to the edge, by Messrs. Barnes and Morris, Ltd., scientific instrument makers, Audrey House, Ely Place, London, E. C. 1]; The University of Edinburgh Mathematical Institute, 87—October 9: Review by S. Brodetsky of T. R. Running's *Empirical Formulas* (New York, 1907), H. B. Phillips's *Differential and Integral Calculus* (New York, 1916–17), W. P. Milne and G. S. B. Westcott's *First Course in the Calculus*, Part 1 (London, 1918), R. C. Fawdry's *Dynamics*, Part 2 (London, 1919), and R. S. Heath's *Solid Geometry* (London, 1919), 109–110; P. E. B. Jourdain, 117 ["With the mathematician Philip Edward Bertrand Jourdain there died on October 1 a truly remarkable man. Jourdain lived only thirty-nine years, but the amount and value of the work that he accomplished, considering the disability under which he labored, are almost incredible. . . . He went up to Cambridge in 1898, then already a cripple. During his course at Cambridge he spent some time in Germany and became a fluent and scholarly linguist, speaking and reading several European languages. In 1904, though now physically quite incapacitated, he was awarded the Allen mathematical scholarship for research, and throughout the remainder of his short career his main activities were directed to the prosecution of mathematical investigations. His most important work was the discovery of certain series of infinite numbers. Working with Russell and Whitehead, he showed that certain arithmetical processes could be applied to them, and thus he obtained new and interesting results. He continued on this line of research, and even a few days before his death, of the imminence of which he was fully aware, he succeeded in demonstrating the existence of a previously unsuspected series of infinities. . . . Jourdain contributed extensive mathematical articles to the last edition of the *Encyclopaedia Brittanica*. He founded and edited the *International Journal of Ethics*. He was for some years the English editor, and since the death of Carus in 1918, the chief editor of the *Monist*. He also made a number of translations for the Open Court Publishing Co. Jourdain took the liveliest interest in the movement for encouraging the history of science. He was a contributor to *Isis*, and at the time of his death he had in preparation an article for the *Studies in the History and Method of Science* which it is hoped he may have left in a state ready for publication"]; "Hindu Spherical Astronomy," 119 ["Mr. G. R. Kaye has published a paper on 'Ancient Hindu spherical astronomy' in the *Journal and Proceeding of the Asiatic Society of Bengal* (Vol. 15). In this he summarises, with the aid of modern mathematical formulae, the fundemantal portions of the principal classical astronomical texts, which date from between A.D. 498 (the Aryabhatiya) and about A.D. 1000, when the redaction of the Surya Sidd-

[1] For this number the *Bulletin* was called *Bulletin of the State Teachers College.*

hanta now extant was written. Indian trigonometry is, like Indian astronomy, of Greek origin, but the Indians developed the methods received from the Greeks in various ways. There seems to be no doubt that the Indians were the first to introduce the use of sines instead of chords, and to compute the table of sines. But they never went further, and did not make use of the tangent function. They never gave a proof of any rule they enunciated. . . ."]

OPEN COURT, volume 33, September, 1919: [a Paul Carus number with portrait] "The ideals of the life and work of Paul Carus" by P. E. B. Jourdain, 521–523 [Quotation: "At the Gymnasium of Stettin he came under the influence of the great mathematician Hermann Grassmann, of whom he always spoke with affectionate respect. Later he studied at the Universities of Strassburg and Tübingen. Owing to the need he felt so strongly for keeping his independence of thought, he resigned a teaching post in Germany and came first to England and then to America"].

REVUE DE MÉTAPHYSIQUE ET DE MORALE, 26. année, July–August, 1919: "A propos de la démonstration géométrique: Réponse à M. Goblot" by L. Rougier, 517–521.

REVUE GÉNÉRALE DES SCIENCES PURES ET APPLIQUÉES, 30. année, nos. 15–16, August, 1919: "La vie et l'oeuvre de Léonard de Vinci. A propos du quatrième centenaire de sa mort" by F. Bottazzi, 465–477; Review by R. D'Adhémar of P. Boutroux's *Les Principes de l'analyse mathématique. Exposé historique et critique*, volume 2 (Paris, 1919), 492.

REVUE SCIENTIFIQUE, August 16–23, 1919: "Le temps et sa mesure" by M. Moulin, 486–494.

SCHOOL SCIENCE AND MATHEMATICS, volume 19, no. 7, October, 1919: "What graphical and statistical material should be included in the ninth-grade mathematics course?" by L. E. Mensenkamp, 595–598; "Developing ability to solve the verbal problem; the basic aim of the ninth grade course" by Elsie G. Parker, 599–604; Problems and solutions, 655–658; "The theorem of Nichomachus" by U. P. Davis, 663 ["Cubical numbers are sums of consecutive odd integers"]; "Central Association of Science and Mathematics Teachers" by C. S. Winslow, 664–665.

SCIENCE, new series, volume 50, August 30, 1919: "New activities in the history of science" by L. C. Karpinski, 213–214—September 12: "Not ten but twelve!" by W. B. Smith, 239–242 [advocating duodenary scale of notation].

SCIENTIA, volume 26, no. 3, September, 1919: "La teoria di relatività nel suo sviluppo storico. Porte I^a: La relatività della prima maniera" by A. Palatini, 195–207, supplément, 59–72; "Le danger de l'application du calcul des probabilités aux sciences de la nature et en particulier à l'astronomie" by E. Belot, 242–246; Review by G. Scorza of Montessus de Ballore's *Leçons sur les fonctions elliptiques en vue de leurs applications* (Paris, 1917), and *Exercices et leçons de mécanique analytique* (Paris, 1915), 247–248; Review by U. Ricci of G. H. Knibbs's The Mathematical theory of population (Melbourne, 1917), 260 [The review: "L'éminent directeur de la statistique officielle de l'Australie a voulu donner une preuve de sa bravoure en réunissant en un gros volume une collection très étendue de formules mathématiques permettant de mesurer les phénomènes démographiques. Les instruments analytiques propres à définir et à décrire la consistance et les variations d'une population dans ses divers aspects (distribution d'une population par sexe et par âge, masculinité, c'est-à-dire proportion entre hommes et femmes, natalité, nuptialité, fécondité, mortalité, migrations) sont forgés, adaptés et appliqués avec grande richesse, nous dirons presque avec luxe. De très nombreuses courbes et surfaces sont dessinées dans le volume, et côte-à-côte avec la théorie s'avancent les exemples tirés du recensement australien de 1911 et d'autres statistiques australiennes, sans parler de comparaisons établies avec les données empruntées à d'autres pays. Tous ceux qui s'occupent de statistique générale et de démographie consulteront avec profit ce volume qui est une mine de précieuses et minutieuses études"].

SCIENTIFIC AMERICAN SUPPLEMENT, volume 88, September 20, 1919: "Derivation of new magic squares" by "Weg," 191.

SCIENTIFIC MONTHLY, volume 9, no. 4, October, 1919: "Linkages" by F. V. Morley, 366–378.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 20, no. 3, July, 1919: "Certain types of involutorial space transformations" by F. R. Sharpe and V. Snyder, 185–202; "On a new treatment of theorems of finiteness" by O. E. Glenn, 203–212; "On the theory of developments of an abstract class in relation to the calcul fonctionnel" by E. W. Chittenden and A. D. Pitcher, 213–233; "On the influence of keyways on the stress distribution in cylindrical shafts" by T. H. Gronwall, 234–244; "Some convergent developments associated with

irregular boundary conditions" by J. W. Hopkins and D. Jackson, 245–259; "Groups possessing a small number of sets of conjugate operators" by G. A. Miller, 260–270.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 50, no. 4–5, April, 1919: "Ein neues elementares Verfahren zur Lösung von Extremaufgaben" by H. Dörrie, 153–177; "Ueber die Konstruction der Ellipse" by E. Wiedemann, 177–181; "Aufgaben-Repertorium," 189–192.

AMERICAN DOCTORAL DISSERTATION

J. H. Weaver, "Some extensions of the work of Pappus and Steiner on tangent circles." AMERICAN MATHEMATICAL MONTHLY, January, 1920, volume 27, pp. 2–11; also reprinted. (Pennsylvania, 1916.)

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I. [*1918*, 33–34, 459; *1919*, 167].

April 18, 1919: "The mathematical theory of investments" by Professor Clinton H. Currier.

May 15: "Mathematics in chemistry" by Esther A. Brintzenhoff '19; "Isosceles trigonometry" by Chauncey D. Wentworth '20; "History of calculating machines" by Mr. W. L. Morden, New England manager of the Monroe Calculating Machine Company. Election of officers. Taking of club photograph.

June 4: Picnic.

The average attendance at the meetings for the year 1918–19 was 47. The officers elected for the year 1919–20 were:

Chairman of Club, Professor Roland G. D. Richardson;

Committee on Program, Professor Raymond C. Archibald, Alice F. Hildreth Gr., Pauline A. Barrows '21, Chauncey D. Wentworth '20, Daniel E. Whitford '20;

Committee on Arrangements, Professor Ray E. Gilman, Frances M. Merriam '20, Constance W. Haley '21, Raymond L. Wilder '20, Marshall H. Cannell '22, Bruce H. McCurdy '22.

THE MATHEMATICS CLUB OF CONNECTICUT COLLEGE, New London, Conn. [*1918*, 270, 460].

Active membership in this club is limited to students pursuing courses in mathematics beyond the regular freshman requirement. There were ten members of the club in 1918–19 and due to influenza and diphtheria quarantines as well as to war conditions only four formal meetings were held.

The officers for 1918–19 were: President, Margaret Maher '19; secretary, Justine McGowan '20; treasurer, Louise Avery '21. These officers constitute

the committee on program and arrangements. At the meeting of May 16, 1919, Justine McGowan '20 was elected president for the year 1919–20.

Programs for the four meetings are given below.

November 23, 1918: "Arithmetic in the educational program of earlier days" by Professor David D. Leib.

February 4, 1919: "Flatland" by Dorothea E. Peck '19.

April 15: "Calculating machines" by Marie Munger '20.

May 16: "Mathematical puzzles and paper cutting" by Dorothy Pryde '21.

Mathematics Club of Iowa State University, Iowa City.

This club was organized and held its first meeting April 3, 1919. During the year 1918–19 it was not primarily an undergraduate club since its officers were all graduate students or members of the faculty and students had no part in the programs presented. However, junior and senior students were asked to attend the meetings, were allowed to vote and an effort was made to keep the matter presented within the range of their mathematical advancement. It is intended that undergraduate students shall take part in the programs to be given in 1919–20.

The officers for the remainder of the year 1918–19 were: President, E. M. Berry Gr.; secretary-treasurer, Helen J. Williams Gr.; program committee, E. M. Berry Gr., Professor Richard P. Baker and Professor Edward W. Chittenden.

The programs for the remainder of the year 1918–19 are given below.

April 3, 1919: "On functional relations for which the correlation coefficient is zero" by Professor Henry L. Rietz.

April 17: "Graphical solutions for the imaginary roots of an algebraic equation" by Rutherford E. Gleason, Instructor in Mathematics.

May 1: "Computing the mean and standard deviations of a frequency distribution" by Professor John F. Reilly.

May 15: "Some applications of differential equations" by Professor Raymond B. McClenon, Grinnell College, Grinnell, Iowa.

May 28: "Some properties of cubic curves" by Frank M. Weida, Instructor in Mathematics.

Mathematics Club of Iowa State Teachers College, Cedar Falls

[*1918*, 311–312, 459].

The officers for the summer term 1918 were: President, Professor Peter Luteyn; secretary, Lorena F. French '18; executive committee, Professor Ira S. Condit, Professor Charles W. Wester and Margery Kinne '21.

The officers for the fall, winter and spring terms were: President, Professor Emma F. Lambert; secretary, Mary A. Peters '19; executive committee, Professor Charles W. Wester, Professor Robert D. Daugherty and Garnet Maulsby '20. The executive committee prepares the programs and decides on the dates of meetings.

Programs between May, 1918, and May, 1919, were as follows:

May 8, 1918: "Group recitations in geometry" by Phoebe Cowan '18; "Teaching of algebra in secondary schools" by Garnet Maulsby '20; "The number concept" by Verna Zarr '18.

May 29: "The correlation of mathematics in secondary schools" by Lorena F. French '18; "Calculus applied to physics" by Laura Huber '19.

June 27: "Tests of efficiency in teaching" by Mr. Charles W. Kline, superintendent of schools, East Waterloo, Iowa.

July 10: "Supervised study in secondary schools" by Miss Jessie Cunning, Instructor in Mathematics, Ft. Dodge High School, Ft. Dodge, Iowa; "The equation as an interpretation of the problem" by Principal W. E. Beck, Iowa City High School.

July 31: "Important points to be emphasized in the teaching of algebra" by Principal W. E. Beck, Iowa City High School.

August 14: "Standard forms in teaching arithmetic" by Miss Olive Tilton, Supervisor of Mathematics, Iowa State Teachers College Training School.

November 20: "An introduction to modern geometry" by Professor Wester; "Pascal's theorem" by Eleanor Sweeney '19; "Brianchon's theorem" by Garnet Maulsby '20; "Harmonic sets" by Dora Hospers '20; "Poles and Polars and applications to metrical relations" by Mary A. Peters '19.

January 29, 1919: "Rhythm in number land" by Professor Daugherty.

February 19: "The development of number" by Garnet Maulsby '20.

March 19: "Chicago meeting of the Department of Superintendence, N. E. A." by Professor Condit; "The fourth dimension" by Eleanor Sweeney '19.

April 16: "The metric system" by Professor Lambert.

April 30: "Fractions" by Professor Wester; "Irrational number" by Bernice Wilcox '19.

MATHEMATICS CLUB OF MOUNT HOLYOKE COLLEGE, South Hadley, Mass. [*1918*, 312–313, 458].

October 19, 1918: Social meeting.

November 16: "The abacus and other forms of counting machines" by Dorothy C. Smith '19.

January 18, 1919: "Fundamental principles of flight" (illustrated with models of monoplanes) by Elizabeth R. Laird, Professor of Physics.

February 15: "History and applications of logarithms and the slide rule" by Thelma Bridge '20, Margaret F. Wilcox '19 and Mildred Allen, Instructor in Physics.

MATHEMATICS CLUB OF THE UNIVERSITY OF WASHINGTON, Seattle [*1919*, 170].

An account of this club was published in the April, 1919, number of the MONTHLY. The supplementary items of information given below are found in the MONTHLY for December, 1905.

The club was organized in the autumn of 1905. The first officers were: President, Professor Robert E. Moritz; secretary, Professor Frank M. Morrison. The monthly meetings of the club were devoted to reviews, current mathematical literature and reports on original work.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about problems and solutions to B. F. FINKEL, Springfield, Missouri.

PROBLEMS FOR SOLUTION.

2799. Proposed by H. C. BRADLEY, Massachusetts Institute of Technology.

A newspaper recently gave this problem: Cut a regular six-pointed star into the fewest number of pieces which will fit together and make a square. The newspaper gave a solution in seven pieces. First cut off two opposite points of the star. Divide each into two parts, and fit to the remaining portion of the star so as to make a rectangle. Find the mean proportional between the length and breadth of this rectangle (construction not shown); this is the side of the required square. Using this dimension on the two long sides of the rectangle, divide the latter into three pieces, which make the square. Total seven pieces.

How may the square be formed with not more than five pieces?

2800. Proposed by A. M. HARDING, University of Arkansas.

If $x + y + z = xyz$, show that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}.$$

2801. Proposed by A. S. HATHAWAY, Rose Polytechnic Institute.

A dog at the center of a circular pond makes straight for a duck which is swimming along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as $n : 1$, determine the equation of the curve of pursuit and the distance the dog swims to catch the duck.

2802. Proposed by WARREN WEAVER, Throop College of Technology.

Consider two circles, each of radius k, with centers at $(0, 0)$ and $(k', 0)$ respectively, where k' is less than k. Through the point $(k', 0)$ draw a ray making an angle θ with the positive x-axis. Call the intersection of this line with the first circle A, and with the second circle B. Extend the line through the point $(k', 0)$ in the opposite direction, and call the intersection of this extension with the first circle A', and with the second circle B'. Prove that the sum of the two segments AB and $A'B'$ is independent of k, and depends only upon k', i. e. the shift of the circles, and θ.

2803. Proposed by S. A. COREY, Des Moines, Iowa.

In the November, 1918, number of the *Proceedings of the Edinburgh Mathematical Society* (Vol. 36, part 2, page 103), Professor Whittaker gives the following formula for the solution of algebraic and transcendental equations:

The root of the equation

$$0 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots,$$

which is smallest in absolute value, is given by the series

$$-\frac{a_0}{a_1}-\frac{a_0^2a_2}{a_1\begin{vmatrix}a_1a_2\\a_0a_1\end{vmatrix}}-\frac{a_0^3\begin{vmatrix}a_2a_3\\a_1a_2\end{vmatrix}}{\begin{vmatrix}a_1a_2\\a_0a_1\end{vmatrix}\begin{vmatrix}a_1a_2a_3\\a_0a_1a_2\\0\ a_0a_1\end{vmatrix}}-\frac{a_0^4\begin{vmatrix}a_2a_3a_4\\a_1a_2a_3\\a_0a_1a_2\end{vmatrix}}{\begin{vmatrix}a_1a_2a_3\\a_0a_1a_2\\0\ a_0a_1\end{vmatrix}\begin{vmatrix}a_1a_2a_3a_4\\a_0a_1a_2a_3\\0\ a_0a_1a_2\\0\ 0\ a_0a_1\end{vmatrix}}-\cdots.$$

In case of any algebraic equation with imaginary or complex roots the above formula clearly fails. State the conditions under which the formula may be relied upon to give correct results.

2804. Proposed by T. H. GRONWALL, Washington, D. C.

Show that for $|x|<1$

$$\frac{1}{\sqrt{1-x^4}}\int_0^x\frac{dx}{\sqrt{1-x^4}}=x+\sum_1^\infty\frac{3\cdot7\cdots(4n-5)(4n-1)}{5\cdot9\cdots(4n-3)(4n+1)}x^{4n+1},$$

$$\left(\int_0^x\frac{dx}{\sqrt{1-x^4}}\right)^2=x^2+\sum_1^\infty\frac{3\cdot7\cdots(4n-5)(4n-1)}{5\cdot9\cdots(4n-3)(4n+1)}\cdot\frac{x^{4n+2}}{2n+1}.$$

2805. Proposed by C. N. MILLS, Brookings, S. Dakota.

Derive the expression for volume

$$v=\int\int\int\rho^2\sin\phi d\rho d\phi d\theta.$$

In Byerly's *Integral Calculus*, page 183, revised edition, is a method by revolution, and in Czubers's *Integralrechnung*, page 200, is a method using the Jacobian determinant.

Required, a simple method one might use in developing the volume integral in polar coordinates.

2806. Proposed by R. E. MORITZ, University of Washington.

An anthropologist told me recently that large numbers of Russian peasants, whose knowledge of numbers is limited to multiplication and division by 2, employ the following method of multiplication which they were taught by a priest.

(1) Write the two numbers to be multiplied in the same horizontal line.

(2) Multiply the first number by 2 and write the product under the number so multiplied.

(3) Divide the second number by 2, discarding the remainder 1 when it occurs, and write the quotient under the number so divided.

(4) Treat the product and quotient thus obtained in the same manner as the original numbers. Continue this process until the quotient 1 is obtained.

(5) Strike out all the numbers on the left for which the corresponding numbers on the right are even.

(6) Add the remaining numbers on the left. Their sum is the required product.

Problem: Prove that this rule is correct.

SOLUTIONS OF PROBLEMS.

442. (Geometry) [1914, 156; 1919, 268]. Proposed by J. B. SMITH, Hampden-Sidney College.

If any three straight lines AD, BE, CF, be drawn from the corners of the triangle ABC to the opposite sides a, b, c, they will enclose an area. If Δ, Δ'' be the areas of the triangles ABC, DEF, show that

$$\frac{\Delta''}{\Delta}=\frac{(AF\cdot BD\cdot CE-AE\cdot CD\cdot BF)^2}{(ab-CE\cdot CD)(bc-AE\cdot AF)(ca-BF\cdot BD)},$$

where the signs of the factors are to be determined by the following rule: Each segment being measured from one of the corners of the triangle ABC, along one of the sides, is to be regarded as positive or negative according as it is drawn towards or from the other corner in that side.

I. Solution by Frank Irwin, University of California.

The proposition is incorrectly stated: D, E, F being the points of intersection with the opposite sides of the lines drawn through the vertices, Δ'' should be the area of the triangle, $A'B'C'$, formed by these three lines (not of DEF).

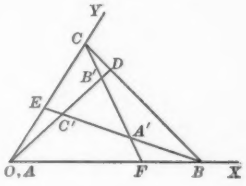
Taking AB, AC as the axes of x, y, $y = mx$ as the equation of AD, and $AE = b_2$, $AF = c_1$, the problem is a straightforward piece of analytic geometry. We readily find that D is the point $[bc/(b + cm), mbc/(b + cm)]$; A', $[(b - b_2)cc_1/(bc - b_2c_1), (c - c_1)bb_2/(bc - b_2c_1)]$; B', $[bc_1/(b + c_1m), mbc_1/(b + c_1m)]$; C', $[b_2c/(b_2 + cm), mb_2c/(b_2 + cm)]$.

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) being given by the expression

$$\tfrac{1}{2}\sin A \cdot \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

we have

$$\frac{\Delta''}{\Delta} = \frac{1}{bc} \cdot \begin{vmatrix} \dfrac{(b - b_2)cc_1}{bc - b_2c_1} & \dfrac{(c - c_1)bb_2}{bc - b_2c_1} & 1 \\ \dfrac{bc_1}{b + c_1m} & \dfrac{mbc_1}{b + c_1m} & 1 \\ \dfrac{b_2c}{b_2 + cm} & \dfrac{mb_2c}{b_2 + cm} & 1 \end{vmatrix}.$$



If we subtract m times the first column from the second, we get

$$\frac{\Delta''}{\Delta} = \frac{1}{bc} \cdot \frac{(c - c_1)bb_2 - m(b - b_2)cc_1}{bc - b_2c_1} \cdot \left(\frac{b_2c}{b_2 + cm} - \frac{bc_1}{b + c_1m} \right) = \frac{[(c - c_1)bb_2 - m(b - b_2)cc_1]^2}{bc(bc - b_2c_1)(b_2 + cm)(b + c_1m)}.$$

On the other hand we find, by similar triangles, that $BD = acm/(b + cm)$; whence $CD = ab/(b + cm)$. Also $BF = c - c_1$, $CE = b - b_2$. A little examination will show that, with the convention as to signs made in the statement of our proposition, these values hold, not only for the case exhibited in the figure, but whatever the positions of the points D, E, F. If, for instance, D lie in BC produced then BD is positive and m negative; but as $b + cm$ is then also negative (for the abscissa of D, $bc/(b + cm)$, is then negative), BD still has the value given above (rather than its negative).

The right side of our proposed equation is now equal to

$$\frac{\left[c_1 \dfrac{acm}{b + cm}(b - b_2) - b_2 \dfrac{ab}{b + cm}(c - c_1) \right]^2}{\left[ab - (b - b_2)\dfrac{ab}{b + cm} \right](bc - b_2c_1)\left[ca - (c - c_1)\dfrac{acm}{b + cm} \right]}.$$

As this reduces at once to the value found above for Δ''/Δ, our proposition is proved.

It may be noted that for the special case where the three lines drawn through the vertices meet in a point we get the familiar property $AF \cdot BD \cdot CE = AE \cdot CD \cdot BF$.

II. History of the Problem and Cesàro's Solution, by R. C. Archibald, Brown University.

The result of this problem, in its corrected form, was given by E. Cesàro in an article published more than twenty-five years ago.[1] It is, however, equivalent to the following, given twelve years earlier by C. Herkema[2]:

$$\frac{\Delta''}{\Delta} = \frac{n^2p^2(m - q)^2}{(mn + mp + np)(mp + np + nq)(np + nq + pq)},$$

where $BD/DC = n/p$, $CE/EA = p/q$, $AF/FB = m/n$. On the special assumption that

[1] "Etude de transversales," *Mathesis*, April, 1884, Volume 4, pp. 85–86.

[2] *Nouvelles annales de mathématiques*, 1872, (2) Volume 11, p. 477.

$AF = k \cdot AB$, $BD = k \cdot BC$, $CE = k \cdot CA$, T. Clausen showed in 1832[1] that

$$\frac{\Delta''}{\Delta} = \frac{1 - 4k + 4k^2}{1 - k + k^2}.$$

In the same number of "Crelle" (page 202) Steiner generalized this result.[2] He showed that if $AF = \alpha \cdot AB$, $BD = \beta \cdot BC$, $CE = \gamma \cdot CA$,

$$\frac{\Delta''}{\Delta} = 1 - \frac{(1-\alpha)\gamma}{1 - \alpha + \alpha\gamma} - \frac{(1-\beta)\alpha}{1 - \beta + \beta\alpha} - \frac{(1-\gamma)\beta}{1 - \gamma + \gamma\beta}.$$

In a problem proposed for solution Cesàro gave yet another form of expression[3] for the ratio of the areas: "Three lines drawn through the vertices of a triangle determine on the opposite sides six segment such that the difference between the product of three non-consecutive segments and the product of the three others is

$$\frac{abc}{a'b'c'}\left(\frac{\Delta''}{\Delta}\right)^2 lmn.$$

In this expression a', b', c' denote the sides of the triangle formed by the three lines; l, m, n the segments of these lines, contained between the vertices and the sides of the first triangle." [4]

The methods of solution of our problem by Steiner and Cesàro are those of elementary geometry. As Cesàro's solution is in a source inaccessible to many MONTHLY readers it is herewith reproduced:

Let $BD = a_1$, $CD = a_2$, $CE = b_1$, $AE = b_2$, $AF = c_1$, $BF = c_2$.

Let us seek first the ratios of the segments determined on each of the lines AD, BE, CF by the two others. The triangles ADC, ADB are cut respectively by the transversals BE, CF, giving

$$\frac{AC'}{C'D} = \frac{ab_2}{a_1b_1}, \qquad \frac{AB'}{B'D} = \frac{ac_1}{a_2c_2};$$

whence

$$\frac{AC'}{AD} = \frac{ab_2}{a_1b_1 + ab_2}, \qquad \frac{AB'}{AD} = \frac{ac_1}{a_2c_2 + ac_1}, \tag{1}$$

Now $a_1b_1 + ab_2 = ab - a_2b_1$, $a_2c_2 + ac_1 = ac - a_1c_2$, consequently,

$$\frac{AC'}{AD} = \frac{ab_2}{ab - a_2b_1}, \qquad \frac{AB'}{AD} = \frac{ac_1}{ca - c_2a_1}. \tag{2}$$

Subtracting one of the equalities (2) from the other

$$\frac{a'}{AD} = \frac{a(a_1b_1c_1 - a_2b_2c_2)}{(ab - a_2b_1)(ca - c_2a_1)}. \tag{3}$$

We have also

$$\frac{\Delta''}{AB'C} = \frac{A'B' \cdot C'B'}{AB' \cdot B'C}, \qquad \frac{AB'C}{ADC} = \frac{AB'}{AD}, \qquad \frac{ADC}{\Delta} = \frac{a_2}{a},$$

from which by multiplication,

$$\frac{\Delta''}{\Delta} = \frac{C'B'}{AD} \cdot \frac{A'B'}{B'C} \cdot \frac{a_2}{a}. \tag{4}$$

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[1] Crelle's *Journal*, 1832, Volume 3, pp. 197–198.

[2] See also Steiner's *Gesammelte Werke*, Volume 1, 1881, p. 166; slips in the result as printed in Crelle's *Journal* are here corrected.

[3] *Nouvelle correspondance mathématique*, Volume 6, March, 1880, pp. 143–144, Question 545.

[4] Neuberg seemed to be the first to show (*Nouvelle correspondance mathématique*, Oct., 1880, Volume 6, p. 472) that

$$\frac{\Delta'}{\Delta} = \frac{a_1b_1c_1 + a_2b_2c_2}{(a_1 + a_2)(b_1 + b_2)(c_1 + c_2)},$$

where $AF/FB = c_1/c_2$, $BD/DC = a_1/a_2$, $CE/EA = b_1/b_2$, and Δ' is the area of the triangle DEF. This result was also stated by Gentry in *Nouvelles annales de mathématiques*, November, 1880, Vol. 39 (2), p. 528.

By means of the equality (3), and analogues to (2) and (3)

$$\frac{CB'}{CF} = \frac{ca_2}{ca - c_2a_1}, \qquad \frac{A'B'}{CF} = \frac{c(a_1b_1c_1 - a_2b_2c_2)}{(ca - c_2a_1)(bc - b_2c_1)},$$

we deduce from (4)

$$\frac{\Delta''}{\Delta} = \frac{(a_1b_1c_1 - a_2b_2c_2)^2}{(bc - b_2c_1)(ca - c_2a_1)(ab - a_2b_1)}.$$

340 (Calculus) [1913, 196; 1919, 213]. Proposed by C. N. SCHMALL, New York, N. Y.

A pencil of parallel rays of light is incident upon a lens whose faces have the radii r_1, r_2, respectively. Show that the distance of the principal focus from the center of the first face of the lens will be a maximum or a minimum when

$$\frac{r_1}{r_2} = \frac{(\mu - 1)^{1/2}}{1 + (\mu - 1)^{1/2}},$$

where μ has its usual meaning.

SOLUTION BY H. S. UHLER, Yale University.

It is necessary to remark at the very beginning that the given result appears incorrect as it does not involve the thickness of the lens, and the statement of the problem is ambiguous since it does not specify whether the first or the second principal focus is meant.

By straightforward analysis, involving Snell's law, but too long to deserve publication in this place, I found

$$x = t + \frac{r_2}{\mu - 1} - \frac{\mu r_2^2}{(\mu - 1)[\mu(r_1 + r_2) - (\mu - 1)t]}, \tag{1}$$

where, to fix the ideas, a double convex lens was contemplated, $x \equiv$ distance from the center of the first (or incidence) face to the second principal focus (the focus beyond the emergence face), $t \equiv$ axial thickness of lens, $\mu \equiv$ index of refraction of the material (glass) of the lens with respect to that of the surrounding medium (air), and r_1 and r_2 denote the arithmetical values of the radii of curvature of the first and second faces of the lens, respectively.

Formula (1) may be checked by referring to James P. C. Southall's *The Principles and Methods of Geometrical Optics*, either edition, page 275, equation (170); and by changing his d, n, and r_2 to t, μ, and $-r_2$, respectively.

Now what quantity is to be the independent variable? Let it be assumed that t is to vary while μ, r_1, and r_2 remain constant. Then the necessary condition for a maximum or a minimum is

$$\frac{dx}{dt} = 1 - \frac{\mu r_2^2}{[\mu(r_1 + r_2) - (\mu - 1)t]^2} = 0,$$

which leads directly to

$$t = \frac{\sqrt{\mu}[r_1\sqrt{\mu} + r_2(\sqrt{\mu} \pm 1)]}{\mu - 1}. \tag{2}$$

To determine which sign has physical meaning we may proceed as follows: $x - t$ must not be negative, otherwise the principal focus will fall inside the material of the lens. Substituting in (1) the expression for $(\mu - 1)\,t$ given by (2) we find

$$x - t = \frac{r_2}{\sqrt{\mu} - 1} \quad \text{or} \quad -\frac{r_2}{\sqrt{\mu} + 1},$$

according as the upper or lower sign in (2) be taken. Since, under all ordinary conditions, $\mu > 1$ the upper sign alone is acceptable.

Continuing,

$$\frac{d^2x}{dt^2} = -\frac{2\mu(\mu - 1)r_2^2}{[\mu(r_1 + r_2) - (\mu - 1)t]^3}$$

which, for the critical value given by (2), reduces to

$$\left(\frac{d^2x}{dt^2}\right)_2 = +\frac{2(\mu - 1)}{r_2\sqrt{\mu}}$$

Since this result is positive the thickness given by (2) corresponds to a minimum focal length, as might be anticipated from purely physical considerations.

It is thus clear that a reasonable interpretation of the statement of the problem leads to an analytical condition which is quite different from the given answer. The thickness of the lens cannot be considered negligible (as is often done in very elementary discussions of problems in geometrical optics) because the numerator of (2) cannot vanish when the upper sign is taken. In other words, t cannot be avoided in the final condition.

In conclusion, attention may be called to the fact that each of $\partial x/\partial r_1 = 0$, $\partial x/\partial r_2 = 0$, $\partial x/\partial \mu = 0$ leads to a result which bears no relation to the given answer and hence t is the only sensible independent variable.

349 (Calculus) [1913, 312]. Proposed by C. N. SCHMALL, New York City.

If $y = a \cos (\log x) + b \sin (\log x)$, eliminate the constants a and b and obtain the equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

SOLUTION BY T. E. MERGENDAHL, Tufts College.

Taking the derivative of y, as given in the problem, and multiplying the result by x, we have

$$x \frac{dy}{dx} = - a \sin \log x + b \cos \log x.$$

Applying the same process to both sides of this equation, we find

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = - (a \cos \log x + b \sin \log x) = - y,$$

or

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

Also solved by W. W. BEMAN, T. M. BLAKSLEE, P. J. DA CUNHA, H. T. DAVIS, H. H. DOWNING, G. H. GRAVES, P. HANSEN, A. M. HARDING, W. L. MISER, A. PELLETIER, ELIJAH SWIFT, and H. S. UHLER.

385 (Calculus) [1915, 161; 1919, 73]. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If $f(x)$ is continuous between a and x, show that

$$\lim_{n=\infty} \frac{\lfloor n}{(x-a)^n} \int_a^x \cdots \int_a^x f(x) dx^n = f(a).$$

T. H. GRONWALL, New York City, offers the following criticism and completion of the solution already published.

First, the passage to the limit for $n \doteq \infty$, which forms the last step (p. 74), requires justification (this is however easily done by using Taylor's theorem with remainder term). Second, the problem as proposed assumes only that $f(x)$ is continuous, so that the use of the *derivatives* of $f(x)$ is not permissible, since they may not exist. The following proof requires $f(x)$ to be bounded in the interval from a to x and to be continuous only at the point a, but not in the whole interval. It is readily shown (integration by parts and complete induction) that

$$\int_a^x \cdots \int_a^x f(x) dx^n = \frac{1}{(n-1)!} \int_a^x (x - \xi)^{n-1} f(\xi) d\xi;^1$$

writing $\xi = a + t(x - a)$ and $f(\xi) = \varphi(t)$, the formula to be proved becomes

$$\lim_{n=\infty} n \int_0^1 (1 - t)^{n-1} \varphi(t) dt = \varphi(0)$$

[1] A proof of this relation may be found in Goursat-Hedrick's *Mathematical Analysis*, Vol. 2, pt. II, p. 36.—EDITORS.

(under the assumption that $\varphi(t)$ is continuous at $t = 0$), or what is the same

$$\lim_{n=\infty} n \int_0^1 (1-t)^{n-1}[\varphi(t) - \varphi(0)]dt = 0.$$

Let $\epsilon > 0$ be as small as we please; $\varphi(t)$ being continuous at $t = 0$, it is then possible to choose a δ, $0 < \delta < 1$, and so small that $|\varphi(t) - \varphi(0)| < \epsilon/2$ for $0 \leqq t \leqq \delta$. Let M be greater than $|\varphi(t) - \varphi(0)|$ for $0 \leqq t \leqq 1$. Then, decomposing the integral into two parts and using the first mean value theorem,

$$\left| n \int_0^1 (1-t)^{n-1}[\varphi(t) - \varphi(0)]dt \right| \leqq \left| n \int_0^\delta \right| + \left| n \int_\delta^1 \right|$$

$$\leqq n \frac{\epsilon}{2} \int_0^\delta (1-t)^{n-1}dt + nM \int_\delta^1 (1-t)^{n-1}dt$$

$$< \frac{\epsilon}{2} + M(1-\delta)^n.$$

Now N may be chosen so large that $M(1-\delta)^n < \epsilon/2$ for $n > N$, so that finally

$$\left| n \int_0^1 (1-t)^{n-1}[\varphi(t) - \varphi(0)]dt \right| < \epsilon \text{ for } n > N,$$

which completes our proof.

192 (Number Theory) [1913, 196; 1919, 214]. Proposed by the late ARTEMAS MARTIN.

Find rational values of v, w, and x that will satisfy simultaneously the conditions

$$(m^2+n^2)(v^2+w^2+x^2)^2 - 4m^2n^2v^2 + m^2n^2(m^2+n^2) = 0,$$

$$(m^2+n^2)(v^2+w^2+x^2)^2 - 4m^2n^2w^2 + m^2n^2(m^2+n^2) = 0,$$

$$(m^2+n^2)(v^2+w^2+x^2)^2 - 4m^2n^2x^2 + m^2n^2(m^2+n^2) = 0,$$

m and n being known rational quantities.

DISCUSSION BY H. S. UHLER, Yale University.

The following analysis shows that the problem is impossible.

Any set of values of v, w, and x that fulfil the given conditions must also satisfy the sum of the three equations, that is, must satisfy

$$3(m^2+n^2)(v^2+w^2+x^2)^2 - 4m^2n^2(v^2+w^2+x^2) + 3m^2n^2(m^2+n^2) = 0.$$

From this equation, we have

$$v^2+w^2+x^2 = \frac{mn}{3(m^2+n^2)}[2mn \pm i\sqrt{9m^4+14m^2n^2+9n^4}].$$

In general, the right hand member is irrational, and the sum of the squares of any number of rational numbers cannot have an irrational value; therefore, the problem is impossible.

We may proceed farther and find explicit expressions for v, w, and x. By inspection, or by forming the differences between the given conditions, taken in pairs, we see that $v^2 = w^2 = x^2$. Consequently,

$$x^2 = \frac{mn}{9(m^2+n^2)}[2mn \pm i\sqrt{9m^4+14m^2n^2+9n^4}],$$

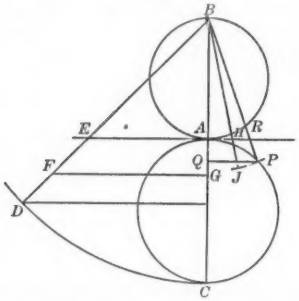
or

$$x = \pm\frac{1}{3}\sqrt{\frac{mn}{2(m^2+n^2)}}\{\sqrt{3(m^2+n^2)+2mn} \pm i\sqrt{3(m-n)^2+4mn}\}.$$

Obviously, these four roots are irrational, in general.

2734 [1918, 444]. Proposed by E. L. REES, University of Kentucky.

Given two circles tangent to each other externally. From the extremity of a diameter through the point of tangency draw a secant such that the segment between the circles shall be equal to a given segment.



SOLUTION BY THE PROPOSER.

Let PB be the required position of the secant, assuming the problem having been solved. Draw PQ perpendicular to AC and let $BP = y$, $AQ = x$, $AB = d$, $AC = d'$, and $RP = a$, the given segment. Then we have, $y^2 = (d + x)^2 + x(d' - x)$. Also $(y - a)/d = (d + x)/y$. Eliminating x, we get $y^2 - 2a'y - d^2 = 0$, where $a' = a(2d + d')/2(d + d')$. Hence,

$$y = a' \pm \sqrt{a'^2 + d^2}.$$

The analysis suggests the following construction: With B as a center, BC as a radius, describe the arc cutting the horizontal diameter produced of the lower circle in D. Draw BD cutting the common tangent in E. Take $EF = a$. Draw FG perpendicular to AC. Then AG will equal a'. Make $AH = AG$. Draw BH and produce it to J, making $HJ = AH$. With B as a center and BJ as radius, describe an arc cutting the lower circle in P. Then BP is the required secant.

Also solved by P. J. DA CUNHA.

2741 [1919, 35]. Proposed by H. L. OLSON, New Hampshire College.

Prove or disprove the following statement: If the three sides and the area of a triangle are integers, at least one of the three altitudes is an integer.

I. SOLUTION BY FRANK IRWIN, University of California.

The statement is not true since the triangle whose sides are 5, 29, 30 has area 72, and altitudes 144/5, 144/29, and 24/5.

II. REMARKS BY J. L. RILEY, Stephenville, Texas.

On page 12 of Carmichael's *Diophantine Analysis* we find the following theorem:

A necessary and sufficient condition that rational numbers x, y, z shall represent the sides of a rational triangle is that they shall be proportional to numbers of the form $n(m^2 + h^2)$, $m(n^2 + h^2)$, $(m + n)(mn - h^2)$, where m, n, h are positive rational numbers and $mn > h^2$.

In deriving this result it is pointed out that if $x = n(m^2 + h^2)$, $y = m(n^2 + h^2)$, and $z = (m + n)(mn - h^2)$, the area is $hmn(m + n)(mn - h^2)$ and the altitude upon z is $2hmn$. If m, n, and h are integers ($mn > h^2$) we have a series of triangles for which the statement of Mr. Olson is true.

2742 [1919, 36]. Proposed by C. N. SCHMALL, New York City.

In Gregory's *Examples in the Differential and Integral Calculus*, 1841, Chap. VII, p. 124, ex. 22, I find the following celebrated problem: "To find a point within a triangle from which if lines be drawn to the angular points their sum may be the least possible." The author remarks that "the direct solution of this problem is long and complicated, etc." Required a simple, brief solution.

I. SOLUTION BY F. V. MORLEY, Johns Hopkins University.

Consider the three points, α, β, γ (complex variable) as on a unit or base circle. Then th sum of the distances from the point x is

$$\sum^{3} \sqrt{(x - \alpha)\left(\bar{x} - \frac{1}{\alpha}\right)} \tag{1}$$

where $x - \alpha = \rho l$, the distance times the turn, or direction factor, and $\bar{x} - 1/\alpha = \rho/l$, the conjugate expression. Since (1) is to be a minimum, D_x and $D_{\bar{x}}$ must vanish. As they are interdependent, either will suffice.

$$D_{\bar{x}} = \sum^{3} \sqrt{\frac{x - \alpha}{\bar{x} - 1/\alpha}} = \sum^{3} l = 0. \tag{2}$$

This means that sum of the three roots (the symmetric function s_1) vanishes in the equation

$$l^3 - s_1 l^2 + s_2 l - s_3 = 0. \tag{3}$$

The condition $D_x = 0$ involves $s_2 = 0$, so that equation (3) reduces to

$$l^3 - s_3 = 0$$

the roots of which are

$$l, \ \omega l, \ \omega^2 l,$$

when ω is a cube root of 1. That is, the three angles at x are all equal to 120°.

The argument may also be presented in its physical, instead of geometrical, aspect. The method in each case is the same.

Consider α, β, γ as centers of attraction for constant forces of equal magnitude. The total potential of x will then be the sum of the distances from x to the three points. For the total potential to be a minimum is to have equilibrium, and the resultant force in any direction must vanish. Hence, the three angles at x are all equal to 120°. It is to be noted that the partial derivative ($D_{\bar{x}}$) of potential, expresses the resultant force with direction factor included.

The well known construction for (the Fermat point) x is to draw on each side $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$, an equilateral triangle, outward, calling the free vertices γ', α', β'. The points γ', α, x, β, are concyclic, and it follows that the line $\gamma' x$ bisects the angle $\alpha x \beta$ and coincides with the line $x\gamma$. x will therefore be the intersection of the three lines $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$.

II. Solution by R. E. Moritz, University of Washington.

Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$, represent the angular points of the triangle expressed in Cartesian coördinates, $P = (x, y)$ any point within the triangle, and θ_i the angle which the line $P_i P$ makes with the positive direction of the x-axis. Then, if D denotes the sum of the distances of P from the angular points, we have

$$D = \Sigma_i \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

In order that D may be a minimum the partial derivatives of D with respect to x and y must be separately equal to zero; hence,

$$\Sigma_i \frac{x - x_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0, \qquad \Sigma_i \frac{y - y_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} = 0,$$

that is,

$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0, \qquad \sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0,$$

or

$$\cos\theta_1 + \cos\theta_2 = -\cos\theta_3, \qquad \sin\theta_1 + \sin\theta_2 = -\sin\theta_3.$$

By squaring and then adding the resulting equations we get $2\cos(\theta_1 - \theta_2) = -1$, from which $\theta_2 - \theta_1 = 120°$. Similarly, $\theta_3 - \theta_2 = 120°$, $\theta_1 - \theta_3 = 120°$.

The problem, therefore, resolves itself into that of finding a point within the triangle at which the three sides subtend equal angles. This point is readily found as follows:

Describe on two sides of the triangle segments of circles containing each an angle of 120°. The intersection point of these circles is the required point. If one of the angles of the triangle is equal to or greater than 120° the intersection point falls on or without the triangle. In that case no point within the triangle satisfies the required conditions.

The following kinematic solution, while perhaps not simpler than the foregoing geometrical solution, has the advantage of being applicable to the more general problem, which I believe has never been solved, namely,

Given n points in a plane, to find a point from which if lines be drawn to the n points their sum may be the least possible.

Draw the triangle to a given scale on a drawing board and insert thumb tacks at the points

marking the vertices. Take a flexible string and fasten one end of it to one of the thumb tacks, say to A. Pass the other end of the string through a ring R, then around the thumb tack B, then back through the ring, then around the thumb tack at C, then once more through the ring R and then back to A. Now pull the free end of the string taut until the ring assumes a fixed position. The center of the ring R will be the required point. The proof is obvious.

III. Note by William Hoover, Columbus, Ohio.

The following extract from a paper by P. G. Tait[1] meets the requirement fully and is of interest historically:

The following problem, originally proposed by Fermat to Torricelli, *To find the point the sum of whose distances from three given points is the least possible*, seems to have given considerable trouble to the older mathematicians, and even in modern times (see *Gregory's Examples*, p. 126) to have been solved in a very tedious manner. Simpler solutions have since been given (*e.g.*, *Cambridge and Dublin Mathematical Journal*, VIII, p. 92), but none, to my knowledge, so direct, as that indicated by Quaternions. The object of this note is to show the simplicity of the quaternion method.

If α, β be the vectors of two of the given points, the origin being the third, and if ρ be the vector of the required point, we must have (by the conditions of the problem)

$$T\rho + T(\alpha - \rho) + T(\beta - \rho) \text{ a minimum.}$$

Hence,

$$S[U\rho - U(\alpha - \rho) - U(\beta - \rho)]d\rho = 0,$$

for all values of $Ud\rho$. Hence the versor sum in square brackets must vanish identically. The immediate interpretation is, that *lines parallel* to ρ, $\rho - \alpha$, $\rho - \beta$ form an equilateral triangle. The required point is therefore in the same plane as the three given points; and their distances, two and two, subtend equal angles at it, which is the well-known solution.

Equally simple is the quaternion solution of the same problem if more than three points be given. Let their vectors, to any origin, be α, β, γ, etc., and let ρ be the vector of the sought point. We have

$$\Sigma \cdot T(\alpha - \rho) = \text{minimum},$$

from which, as above, $\Sigma U(\alpha - \rho) = 0$.

Hence, *if unit forces act at the required point, in the lines joining it with the given points, these forces are in equilibrium.* Or, in another form, *a closed equilateral gauche polygon may be drawn whose sides are parallel to the lines joining the sought point with the given ones.*"

IV. Solution by Otto Dunkel, Washington University.

Suppose that at P, a point inside the triangle ABC, we have a minimum with $AP = r_1$, $BP = r_2$, $CP = r_3$. Then r_1 must be the minimum distance from A to the ellipse with foci B and C and passing through P, since for any point Q on the ellipse $BQ + CQ = r_2 + r_3$. But the shortest distance from an external point to a closed convex curve is orthogonal to the curve and we know from the properties of the ellipse that r_1 in this case must make equal angles with the focal rays r_2 and r_3. Applying the same argument to r_2, we see that r_1, r_2, r_3 must make equal angles with one another.

V. Remarks and Historical Notes by R. C. Archibald, Brown University.

This problem has been already discussed in the Monthly by Professors Jackson[2] and Johnson.[3] Professor Jackson's discussion is about the same as Gregory's. The analytic treatment by Professor Moritz is practically identical with that given in 1853 by A. Cohen[4] and in 1902 by E.

[1] *Proceedings of the Royal Society of Edinburgh*, Vol. 6 (1866–69), 1869, pp. 165–166; also in P. G. Tait, *Scientific Papers*, Vol. 1, 1898, pp. 76, 77.

[2] Volume 24 (1917), pp. 42–44.

[3] Volume 24 (1917), pp. 243–244.

[4] *Cambridge and Dublin Mathematical Journal*, Vol. 9, p. 92.

Goursat,[1] and his kinematic solution is given in the latter part of Tait's paper of 1867 from which Professor Hoover quoted. Tait concludes: "This kinematical process, equally with the quaternion one whose form directly suggests it, gives easily the solution of the more general problem,—To find a point such that m times its distance from A, together with n times its distance from B, etc., may be a minimum." (For three points this is solved in Stegmann–Kiepert, *Lehrbuch der Differentialrechnung*, 7. Aufl. 1895, p. 260.) The problem of finding a point in a plane the sum of whose distances from any number of given points is a minimum was solved by Tédenat in *Annales de mathématiques pures et appliquées* (1810–11), pp. 285–291.

The celebrated problem of our question was formulated by Fermat[2] in the seventeenth century. According to Viviani[3] he suggested it to Torricelli (before 1648, for Torricelli died in 1647). Torricelli discovered three solutions, one by "plane loci" (that is, with ruler and compasses), two others by "solid loci" (that is, by means of conic sections). He afterwards proposed it to Viviani in the following terms: "A triangle, each of whose angles is less than one-third of four right angles[4] being given; to find a point from which, if straight lines be drawn to its three angles, their sum shall be a minimum."

Viviani solved the problem after repeated efforts (he says, non nisi iteratis oppugnationibus tunc nobis vincere datum fuit) and his solution is given in the appendix to his *Geometria divinatio*.[5]

The problem was treated by Thomas Simpson[6] in 1750. He gave the following construction for determining the Fermat point: Describe on BC a segment of a circle to contain an angle of 120°, and let the whole circle BCQ be completed. From A to Q_1 the middle point of the arc $BA'C$ draw AQ intersecting the circumference of the circle in P, which will be the point required. Because of this construction Simson has been credited with the theorem: If on the sides of a triangle ABC, equilateral triangles $A'BC$, $B'CA$ and $C'AB$ be described externally AA', BB' and CC' are concurrent[7]—a construction which Mr. Morley cites above.

Simpson treats also the more general problem: Three points A, B, and C being given, to find the position of a fourth point V, so that if lines be drawn from thence to the three former, the sum $a \cdot AP + b \cdot BP + c \cdot CP$, where a, b, c denote given numbers, shall be a minimum.

The further history of generalizations is very extensive.

Also solved by P. J. Daniel, W. W. Gorsline, R. A. Johnson, H. M. Roeser, J. Rosenbaum, L. Weisner and the Proposer.

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[1] *Cours d'analyse mathématique* tome 1; English ed. by Hedrick, 1904, pp. 130–131. See also G. Humbert, *Cours d'Analyse*, tome 1, 1903, pp. 193–196.

[2] *Oeuvres*, Tome 1, Paris, 1891, p. 153: "Datis tribus punctis, quartum reperire, a quo si ducantur tres rectæ addata puncta, summa trium harum rectarum sit minima quantitas." See also Tome 3, 1896, p. 136.

[3] V. Viviani, *De maximis et minimis geometria divinatio*. Florentiæ, MDCLIX, p. 144.

[4] This careful statement is necessary for constructions indicated in solutions above. If one of the angles, A say, is equal to, or greater than, 120°, in a certain sense its vertex is to be considered as the minimum point; compare the discussion of this by Goursat, *l.c.*, by Bertrand in *Journal de mathématiques pure et appliquées*, tome 7 (1843), pp. 155–160, and by Sturm in *Journal für die reine und angewandte Mathematik*, Vol. 97 (1884), p. 51.

[5] *L.c.*, pp. 145–150. Viviani's method of solution is reproduced in D. Cresswell's *Maxima and Minima*, Cambridge, 1812, pp. 120–121; second edition, 1817, pp. 121–122.

[6] *Doctrine and Applications of Fluxions*, London, 1750, § 36.

[7] The first formulation of the result as here stated seems to have been by T. S. Davies in the *Gentleman's Diary* for 1830, p. 36. It is here shown also that $AA' = BB' = CC'$. Problems closely related to this are frequently published, *e.g.*, in *School Science and Mathematics*, Feb., 1918, pp. 170–171; Jan., 1919, pp. 86–87; April, 1919, pp. 374–375.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. J. S. MILLER has resigned the chair of mathematics at Emory and Henry College to accept a similar position at Hampden-Sidney College.

Dr. W. W. KUSTERMAN has been appointed associate in mathematics at the University of Illinois.

Dr. E. P. LANE has resigned his position as instructor in Rice Institute to accept an appointment as assistant professor of mathematics at the University of Wisconsin.

Mr. J. R. OSBORN and Mr. W. E. ARMENTROUT have been appointed assistants at the University of Kentucky.

Dr. T. F. HOLGATE, of Northwestern University, has resumed his duties as professor of mathematics, after serving as Acting President of the University for the last three years. Dr. HOLGATE was honored with the degree of Doctor of Laws by Queen's University at the time of the installation of a new principal and chancellor on October 16.

Dr. C. H. YEATON has resigned his position as instructor in mathematics at the University of Minnesota to accept an appointment as professor of mathematics at the School of Engineering of Milwaukee.

Professor P. P. BOYD, of the University of Kentucky, has been elected president of the Kentucky academy of science.

Since his discharge from the national service, Mr. J. E. DOTTERER has been appointed professor of mathematics and physics in Manchester (Indiana) College.

At the University of Manitoba, Major N. B. MACLEAN, after active service in command of the Royal Garrison Artillery in France during the late war, for which he was awarded the Distinguished Service Order, was recalled last spring to take the directorship of the returned soldiers' tutorial course during the summer. He has now resumed his duties as head of the department of mathematics and astronomy. He has been elected a fellow of the Royal Astronomical Society of Canada of which Professor H. R. KINGSTON, in the same university, was elected president in 1919. Major N. R. WILSON, who also was in service in France, has now returned to his duties as professor of mathematics.

Dr. C. P. SOUSLEY, of the Pennsylvania State College, has been promoted from assistant professor to associate professor of mathematics.

Dr. FRANK SCHLESINGER, director of the Allegheny Observatory of the University of Pittsburgh, has been elected director of the Yale Observatory. Owing to the death of Mr. E. M. Reed, there is released for the general purposes of the Yale Observatory one-third of his estate, which will add to Observatory funds an estimated $60,000 during the year.

Professor F. W. BEAL, of the University of Tennessee, has been appointed assistant professor of mathematics at the University of Pennsylvania.

At Cornell University, Dr. F. W. REED of the University of Illinois has been appointed instructor in mathematics. Mr. H. M. LUFKIN, Mr. H. PORITSKY, and Mr. H. A. STURGES have been appointed assistants.

Dr. R. J. T. BELL, of the University of Glasgow, has been appointed to the chair of pure and applied mathematics in the University of Otago, New Zealand. It will be recalled that Dr. Bell is the author of *An Elementary Treatise on Coördinate Geometry of Three Dimensions*, editor of the fourth edition of Frost's *Curve Tracing* [*1919*, 201–202], and an editor of recent volumes of the *Proceedings of the Edinburgh Mathematical Society*.

Professor A. B. NELSON, of Central University, Danville, Ky., died in November, 1918.

Rev. C. J. BORGMEYER, professor of mathematics and astronomy in St. Louis University since 1896, died December 6, 1919, aged 58. He was one of the charter members of the Association.

Dr. L. G. WELD, director of the Pullman (Ill.) Free School of Manual Training since 1911, died at Chicago, November 29, 1919, as the result of an infected tooth. He was in his fifty-seventh year. Dr. Weld taught mathematics for about twenty-five years at the State University of Iowa where he was professor of mathematics and astronomy and head of the department 1889–1911, dean of the graduate college 1900–1907, director of the school of applied science 1903–1905, and dean of the college of liberal arts 1907–1910. He was the author of *A Short Course in the Theory of Determinants* (1893), of the section on *Determinants* in *Higher Mathematics* edited by Merriman and Woodward (1896), afterwards reprinted as a separate book, and of several pamphlets dealing with the history of Iowa.

Dr. G. L. DEMARTRES, professor of mathematics at the University of Lille, died in July, 1919, aged seventy-one years. His *Cours de Géométrie Infinitésimale* (Paris, 1913) is familiar to many Americans. A Paris announcement of his death concludes: "Il avait subi avec courage l'occupation allemande et contribué par son exemple à maintenir le fonctionnement du haut enseignement."

Nature reports the death on October 5 of Mr. G. W. PALMER who was appointed senior mathematical master and master of the Royal Mathematical School at Christ's Hospital in September, 1911.

Mr. P. E. B. JOURDAIN, of Fleet, England, died on October 1; he was within a few days of being forty years of age. The following passage from a letter dated September 5, and addressed to the Editor-in-Chief of the MONTHLY, seems to indicate a premonition of the approaching end:

"You may possibly be interested to know that Mittag-Leffler has suggested the publication of some of my correspondence with Cantor on well-ordering and other subjects. It is rather an interesting point to me because I have just succeeded in well-ordering any aggregate, and a proof will appear shortly in *Acta Mathematica*. If you are at all interested in the subject . . . there is no one to whom I would rather communicate the more or less interesting facts about this correspondence, and as a matter of fact my health is a rather doubtful quantity so that possibly I may not be able to finish up these things myself."

The Pontécoulant Prize of the Paris Academy of Sciences has been awarded to Professor A. S. EDDINGTON of England for his astronomical researches.

Sir OLIVER LODGE, the British physicist, is to deliver three lectures in Boston January 15, 20, and February 5. His topics are to be "The Reality of the Unseen," "The Evidence for Survival," and "The Destiny of Man."

At the meeting of the Northeast Missouri Teachers Association at Kirksville, Missouri, on October 24 Professor E. J. WILCZYNSKI, of the University of Chicago, gave an address on "The Value of Mathematics" and "The Teaching of Algebra," and Superintendent P. P. CALLAWAY, of Moberly, spoke on "Mathematics as viewed by a superintendent and inspector."

At a meeting of the Association of Teachers of Mathematics in New England, at Boston University on December 6, the following papers were read by members of the Mathematical Association of America: "Modern Geometry as a stimulus to the high school teacher" by G. L. WAGAR of Mount Hermon Boys' School; "The two edged ruler" by J. L. COOLIDGE of Harvard University.

At a meeting of the National Academy of Sciences in New Haven on November 12, 1919, Mr. J. K. WHITTEMORE, of Yale University, presented (by invitation) a paper entitled, "The starting of a ship under constant power."

Professor VITO VOLTERRA, of the University of Rome, delivered six lectures on the Hitchcock foundation at the University of California October 13–21. Four of these lectures dealt with "The propagation of electricity in a magnetic field," and two with "Derivative functional equations." During the week November 3–10 Professor Volterra delivered at the Rice Institute a course of three lectures on "Functions of Composition and a public lecture on "The international organization of science during and after the war." At the University of Illinois November 12–18 he lectured on "Integro-differential equations" and at the University of Chicago, November 19–21 on "Organization of science during

and after the war" and "The propagation of electricity in a magnetic field." On the same dates Professor GEORGE DE BOTHEZAT delivered lectures on, "The fundamental principles of dynamics" and "The present state of the theory of fluid resistance." Professor Volterra has returned to Italy.

At the Brussels meeting of the National Research Council several International Unions were formed. For other Unions, where the representation was insufficient Divisional Committees were formed. In Mathematics a Divisional Committee was nominated and charged with the duty of circulating a draft of Statutes so as to obtain the opinion of the principal bodies concerned. C. J. de la VALLÉE POUSSIN was elected President, and the following four gentlemen as Secretaries of this Committee: Th. De DONDER (Brussels); G. KOENIGS (Paris); M. PETROWITCH (Belgrade), V. REINA (Rome).

We have referred elsewhere (*1920*, 25) to the Journal of the Mathematical Association of Japan for Secondary Schools. The officers of the Association are as follows: President, Professor T. HAYASHI of the College of Science, Tôhoku Imperial University, Sendai; Vice Presidents, M. MIMORI, honorary professor in the Tokyo Higher Technical School, and M. KUNIYEDA, professor of mathematics in the Tokyo Higher Normal School; Secretary-Treasurers, I. MORI, professor of mathematics in the Tokyo Higher Normal School for Girls, and Miss K. HORIGUCHI, teacher of mathematics in the Higher Girls' School attached to the last named normal school.

Since the beginning of 1919 the editor of the *Jahrbuch über die Fortschritte der Mathematik* has been Dr. L. LICHTENSTEIN. The publishers, Vereinigung wissenschaftlicher Verleger Walter Gruyter u. Co., Berlin (formerly the firms: Göschen, Guttentag, Reimer, Trübner, and Veit), expect to issue two large volumes, the one for the years 1914–1915, the other for the years 1916–1917–1918. (See *1919*, 89, 103.)

Concerning Edinburgh University *Nature* makes report as follows: "Since 1914 the department of pure mathematics has occupied a separate building, the Mathematical Institute. . . . This building contains lecture-rooms large and small, mathematical laboratory, reading-room with students' library, research-room with a library of mathematical periodicals and advanced works, and rooms for the staff. The laboratory course comprises interpolation, construction of mathematical tables, numerical solutions of algebraic and transcendental equations, numerical integration, least squares, graduation or adjustment, fitting of normal and skew frequency curves, correlation, practical Fourier analysis, spherical harmonic analysis, periodogram analysis, with drawing-board work in nomography, descriptive geometry and cartography.

"The most recent development is the institution of a diploma in actuarial mathematics. This is intended for students who are employed in the numerous life insurance offices in the city of Edinburgh, and are, therefore, only part-time

students of the University, attending, however, day, not night, classes. The course which covers two years, is conducted on the mathematical side by Professor E. T. WHITAKER, and on the actuarial side by Dr. A. E. SPRAGUE, president of the faculty of actuaries. . . .

"The Mathematical Institute is the meeting-place of the Edinburgh Mathematical Society, and houses the library of the Society.

"In the department of applied mathematics, which is under the charge of Dr. C. G. KNOTT, arrangements have been made for the inclusion of special honor courses on wave-motion in matter and æther, kinetic theory of gases, and radiation. The former courses on dynamics, hydrodynamics, and elasticity have also been extended, one of the full-year courses being specially adapted to the needs of the student of engineering. There is also a post-graduate course on quaternion vector analysis."

The technical staff of the United States Ordnance Department has been authorized to secure the services of five experts in mathematics and dynamics, at salaries ranging from $2,000 to $5,000, to conduct scientific research on ordnance problems, act as advisers on all mathematical and scientific problems, for the ordnance department, and keep up connections between the department and the scientific world.

Increases in Salaries of College Teachers is the title of "Higher Education Circular No. 15" issued by the Bureau of Education, Washington, in July, 1919. Many interesting facts are given.

At its recent Anniversary Meeting the Royal Society of London awarded the Sylvester Medal to Major PERCY ALEXANDER MACMAHON whose

researches on the combinatory analysis and on subjects allied to the partition of numbers are of the highest value, and display great originality and invention. He has shown equal power in the discovery and treatment of the wonderful ranges of partition theorems which are desirable from the theory of elliptic functions, and of similar theorems to be obtained by the application of analysis to purely arithmetical principles.

At the same meeting of the Royal Society a Royal Medal was awarded to Mr. JAMES HAYWOOD JEANS who

has successfully attacked some of the most difficult problems in mathematical physics and astronomy. In the kinetic theory of gases he has improved the theory of viscosity, and, using generalized co-ordinates, has given the best proof yet devised of the equipartition of energy and of Maxwell's law of distribution of molecular velocities assuming the validity of the laws of Newtonian dynamics. In dynamical astronomy he took up the difficult problem of the stability of the pear-shaped form of rotating, incompressible, gravitating fluid at a point where Darwin, Poincaré, and Liapounoff had left it, and obtained discordant results. By proceeding to a third order of approximation, for which very great mathematical skill was required, he showed that this form was unstable. He followed this up by the discussion of the similar problem when the fluid is compressible, and concluded that for a density greater than a critical value of about one-quarter that of water the behaviour is generally similar to that of an incompressible fluid. For lower densities the behaviour resembles that of a perfectly compressible fluid, and with increasing rotation matter will take a lenticular shape and later be ejected from the edge.

CONTINUITY IN SYNTHETIC GEOMETRY.

By JOHN MATHESON, Queen's University.

(Read before the Mathematical Association of America, September 5, 1919.)

1. The geometry of Euclid was a geometry of particular cases. Its main characteristics were the logic of its argument and the sequence of its propositions. The relationship between one theorem and another was only that of logical dependence; and there was no attempt to find general principles which might underlie the separate theorems. The same was true of the later geometry of the conic sections.

A certain amount of generalization has now been attained through the principle of continuity as suggested first by Kepler and developed more fully in the projective geometry of Poncelet. This principle involved two ideas,—1st. the idea of infinity which led to that conception of the conic sections in which the parabola occupies a boundary position between the ellipse and the hyperbola, and which made possible more general statements of projective properties; and 2nd. the idea of imaginary points which led to the generalization of theorems to include the cases in which some of the elements involved are imaginary.

The principle of continuity as thus developed has resulted in much that is distinctive in modern geometry. But very little has been done to examine the extent to which the application to elementary geometry of the wider notion of continuity, as applied to mathematical functions in general, will result in the discovery of greater generalization. Work along this line will be found valuable in the teaching of synthetic geometry.

2. The following definitions will serve as a basis:

(i) A plane curve C is the locus of all the possible positions which a variable point P may take, in a definite order, according to some given law.

(ii) Let A and B be two fixed points in C. Then P is said to *move* from A to B when it takes all its positions, in order, from A to B. In such a case P is said to *generate* the segment AB of C.

It will be seen that the movement, or motion, of P, as thus defined, has not necessarily all the characteristics of physical motion.

(iii) Let L be a fixed point in the plane. If the distance PL becomes less than any assigned value, however small, as P moves in C from whatever direction, then L is the *limit*, or the *limiting point*, of the positions of P as it approaches L. (Cf. Pierpont, *Real Variables*, vol. I, § 254.)

Various cases of this may be illustrated as follows: Let AB be a line segment 4 units long, D its middle point, E any other point on it, and P a variable point on it.

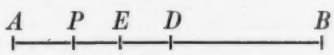

FIG. 1.